PHYS1501 Physics for Engineers I

Joshua Swaim

UConn, Fall 2024

Contents

1 Vectors and Units

1.1 Vector Definition

A **vector** is a magnitude (also called "size" or "length") together with a direction.

1.2 Vectors: Direction

Direction is always given relative to a reference direction. The usual way to specify a reference direction in physics is through perpendicular axes, as shown in Figure 1. If I just say "60°," this is *not* a direction. Even though there's an angle, there is no reference direction specified. If I say, "60° above the x-axis," this *is* a specific direction.

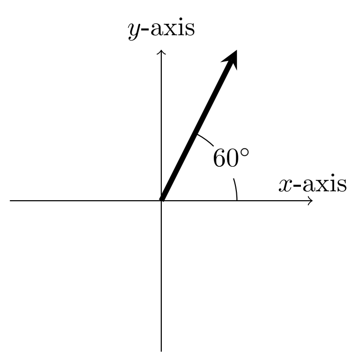


Figure 1: 60° above the x-axis is a specific direction.

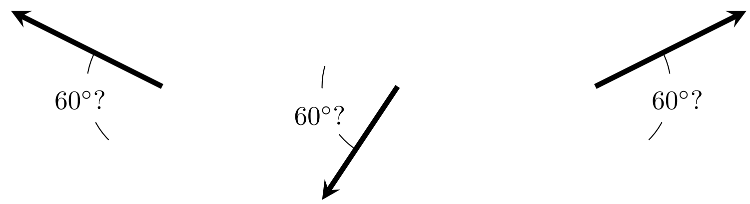


Figure 2: Directions are relative. Saying “60° is not enough to give a direction.

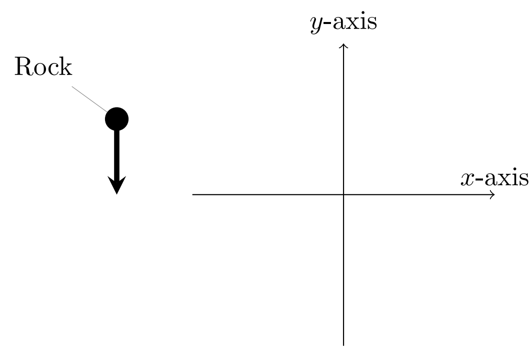
Remember: Direction is relative. To specify a direction, you always need to start with some reference directions. Usually, the reference directions are the x, y, and z axes.

1.2.1 One-dimensional Problems

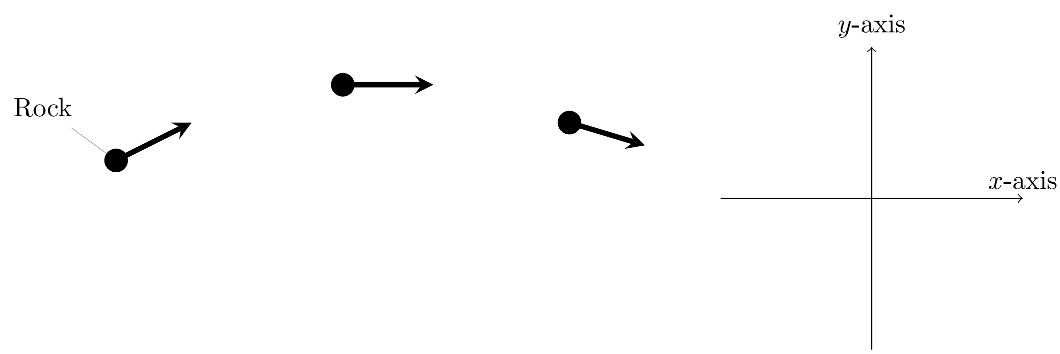
In one-dimensional problems, vectors become much simpler. Remember that a vector is both a magnitude and a direction. In a one-dimensional problem, however, there are only two possible directions: forward and backward. You still need an axis to define which way is forward and which way is backwards. Once you have that, vectors can be described by a single number (along with some units!). The direction of the vector (forward or backward) is given by the sign of the number. Negative numbers indicate a vector that points backwards (relative to your axis) and positive numbers indicate a vector that points forward.

1.2.2 Choosing Axes

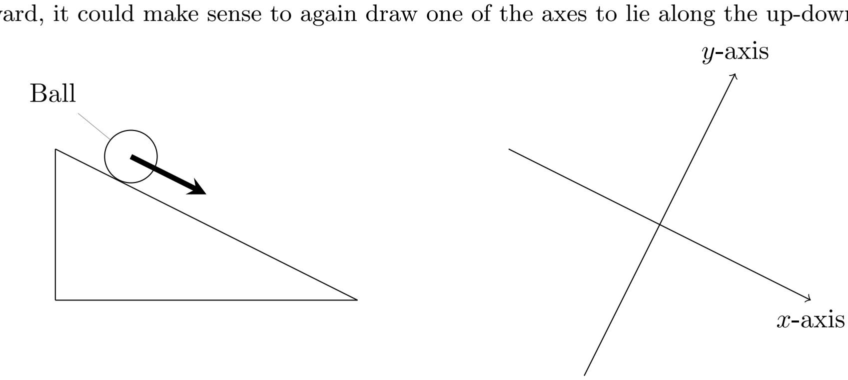
We can choose the axes to point in any direction we want (as long as they are perpendicular to each other). It's usually a good idea to choose them so that they point along directions that are important for the problem. Figure 3 shows examples of good choices of axes for several different problems.



(a) Since the rock is falling down, it makes sense to draw one of the axes to lie along the up-down direction.



(b) Since the rock is changing direction as it moves, it is hard to choose how to draw the axes. Since gravity pulls downward, it could make sense to again draw one of the axes to lie along the up-down direction.



(c) For the ball rolling down the ramp, we could choose one of the axes to lie along the direction of the ramp. Since gravity points downward, we could also choose to draw one of the axes along the up-down direction. In this case, I chose the first option.

Figure 3: Answers for the "Choosing Axes" activity.

1.3 Magnitude

Magnitude is relative. Numbers always need interpretations. **Units** are one way to give interpretations to numbers. Here are some examples demonstrating this point.

- "I walked 5." This makes no sense. The correct way to communicate distance would be something like "5 kilometers" or "5 centimeters".
- "I drove my car at a speed of 120." This also makes no sense. It should be something like "120 miles per hour" or "120 kilometers per second."

Remember: Magnitude is relative. To specify a vector, you always need to give an interpretation to your numbers by giving them units.

1.3.1 Dimension versus Unit

Dimension is the physical quantity being measured. For example: distance, time, and mass are all dimensions. *Units* are a specific way to measure the quantity. For example: meters, miles, and inches are different units with same dimension (length).

1.3.2 Examples of Fundamental Units

In this course, we will mostly use a set of units called SI units. The SI unit for length is the meter. The SI unit for time is the second. The SI unit for mass is the kilogram (*not* the gram).

1.3.3 Prefixes

Sometimes the standard units are too big or too small to be useful. For example, if I tried to give the distance to New York City in meters, I would need to use a very large number. It would be better to give the distance in kilometers. 1 kilometer is defined as 1000 meters. As another example, if I tried to give the width of a hair in meters, I would need to use a very small number. It would be better to give the distance in micrometers. 1 micrometer is defined as 1/1,000,000 of a meter.

These prefixes can be applied to any SI unit. For example, a kilosecond (abbreviated "ks") is 1000 seconds, and a microgram (abbreviated μg) is 1/1,000,000 of a gram. Figure 4 is a table of common SI unit prefixes.

Prefix	Abbreviation	Relationship to Original Unit
nano-	n	$\times 10^{-9}$
micro-	μ	$\times 10^{-6}$
milli-	m	$\times 10^{-3}$
centi-	c	$\times 10^{-2}$
kilo-	k	$\times 10^{3}$
mega-	M	$\times 10^{6}$
giga-	G	$\times 10^{9}$

Figure 4: Common SI unit prefixes.

1.3.4 Derived Units

In addition to fundamental SI units, we can also get more types to units by combining other SI units. For example, the *average* speed of an object is defined by the formula

$$\text{average speed} = \frac{\text{distance traveled}}{\text{time traveled}}.$$

We already know units for distance and time. Suppose I want to calculate the average speed of a car that traveled 6m (”m” stands for meters) in 2s (s stands for seconds). By the above formula

$$\text{average speed} = \frac{6\text{m}}{2\text{s}} = 3\ ?.$$

The units for 3 are m/s (read “meters per second”). The “per” basically means “divided by”.

1.3.5 Unit Conversion

Often, a magnitude will need to be converted from one unit to another. For example, imagine we have a length of

$$5 \text{ miles}$$

We want to know the magnitude of the physical distance in kilometers. 1 mile is the same physical distance as 1.6 kilometers. We could write

$$1 \text{ mile} = 1.6 \text{ kilometers}$$

Therefore, we set up a fraction with 1.6 kilometers on top and 1 mile on the bottom. Because it has the same distance on the top and bottom, this fraction is “equal to 1”.

$$\frac{1.6 \text{ kilometers}}{1 \text{ mile}} = \text{"1"}.$$

Because this fraction is "equal to 1," we can multiply our original distance by this fraction without changing the physical distance:

$$5 \text{ miles} = 5 \text{ miles} \cdot \frac{1.6 \text{ kilometers}}{1 \text{ mile}}$$

Units in equations cancel just like variables.

$$5 \cancel{\text{miles}} \cdot \frac{1.6 \text{ kilometers}}{1 \cancel{\text{mile}}} = 5 \cdot \frac{1.6 \text{ kilometers}}{1} = 5 \cdot 1.6 \text{ kilometers} = 8 \text{ kilometers}.$$

This method can be used even for more complicated unit conversions. Each unit can be converted by multiplying by a fraction "equal to 1."

1.3.6 Dimensional Analysis

Keeping track of units when doing a calculation can help prevent algebra errors. For example, suppose I was trying to calculate a distance, and I ended up with an expression like

$$\frac{1}{2} \cdot \frac{12\text{kg}}{6\text{kg}} (5\text{m/s})^2 \cdot 3\text{s}$$

If we cancel out the units on the top and bottom, we get

$$= \frac{1}{2} \cdot \frac{12\cancel{\text{kg}}}{6\cancel{\text{kg}}} \cdot 25\text{m}^2/\text{s}^2 \cdot 3\text{s} = \frac{1}{2} \cdot \frac{12}{6} \cdot 25\text{m}^2/(\text{s} \cdot \cancel{\text{s}}) \cdot 3\cancel{\text{s}}$$

$$= \frac{1}{2} \cdot 2 \cdot 25\text{m}^2/\text{s} \cdot 3 = 75\text{m}^2/\text{s}$$

I know there must have been an error in my formula, because the units of my final answer are m^2/s, which is not a unit of distance.

Another important thing to remember is that you should never add or subtract magnitudes with different units. For example, "5 miles + 2 seconds" makes no sense. This may seem obvious now, but it's easy to forget when your in the middle of a calculation with more complicated units.

Remember: Always include units in your calculations, and check whether they make sense.

1.4 Working with Vectors

1.4.1 Equivalent Vectors

Vectors have magnitude and direction. These are the only things that matter when specifying a vector. Note that it doesn't matter *where* a vector is drawn. Only its length and direction matter. For example, the vectors in Figure 5 are all equivalent. Even though they are drawn at different locations, they have the same length and direction.

Remember: The location of a vector doesn't matter. Only its magnitude and direction matter.

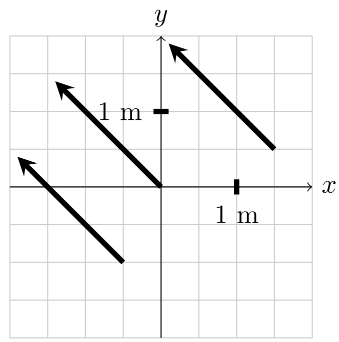


Figure 5: These vectors are all equivalent. Even though they are drawn at different locations, they have the same length and direction.

.

1.4.2 Cartesian Coordinates

Given a set of axes, another way to specify a vector is to give a set of Cartesian coordinates. Below are three methods for getting the coordinates of a vector. If you find this process confusing, I recommend sticking with Method 1.

Method 1 This method is illustrated in Figure 6.

- Draw (dashed) lines through tail of vector, parallel to the axes.
- Find the angle of the vector **above the x-axis**.
- Then $v_x = |\vec{v}| \cos\theta$ and $v_y = |\vec{v}| \sin\theta$.
- Make sure to check the signs of your coordinates. The coordinate should be positive if it lies on the positive side of the axis, and negative otherwise.

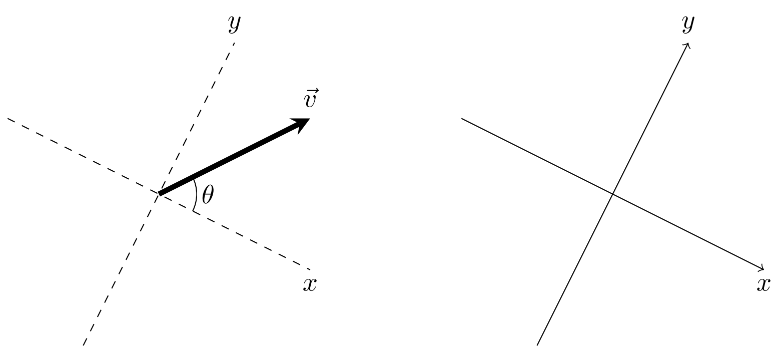


Figure 6: The steps to find the Cartesian coordinates of a vector (method 1).

Method 2 This method is illustrated in Figure 7.

- Draw a set of axes passing through the "tail" end of the vector.
- Draw lines from the end of the vector to each of the axes. Draw them so that they intersect each axis at a 90° angle.
- Use trigonometry to find the length of the portion of each axis between the tail of the vector and the point of intersection.
- These lengths are the Cartesian coordinates of the vector. The length along the x-axis is the x-coordinate, the length along the y-axis is the y-coordinate, etc.
- The coordinate should be positive if it lies on the positive side of the axis, and negative otherwise.

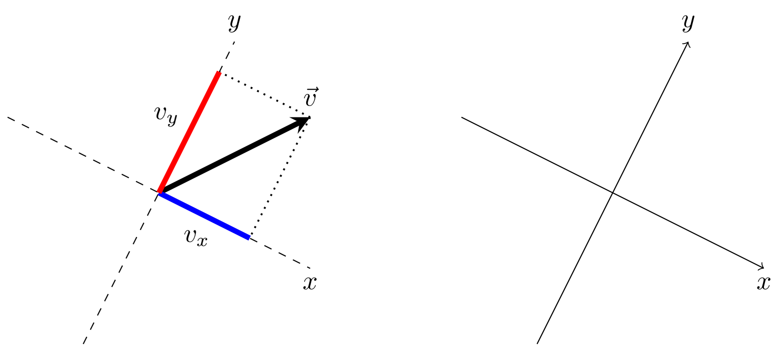


Figure 7: The steps to find the Cartesian coordinates of a vector. The dashed lines are drawn through the tail of the vector and are parallel to the axes. The dotted lines are drawn from the end of the vector and meet the dashed axes at 90°. Each of the dotted lines form a right triangle with one of the axes and the vector as the hypotenuse. The lengths of the blue and red sections give you the x and y coordinates respectively. If you can figure out one of the angles, you can use trigonometry to find the lengths. To figure out whether each coordinate is positive or negative, just look at whether the vector is going in the same direction (+ sign) or opposite direction (- sign) to the axis. In this case, both coordinates are positive because the vector is closer to pointing along the positive x and y directions.

Method 3 This method is illustrated in Figure 8.

- Draw a set of lines parallel to the axes. Draw them so that they form a right triangle with the vector.
- Find the length of the vector (usually, this is already known). This gives you the length of the hypotenuse of the right triangle.

- Find one of the angles. You might need to use some trigonometry.
- Use sin and cos functions to find the lengths of the two legs of the triangle. These give you the absolute value of the x and y coordinates of the vector.
- To figure out whether each coordinate should be positive or negative, just look at whether the vector is going in the same direction (+ sign) or opposite direction (- sign) to the axis.

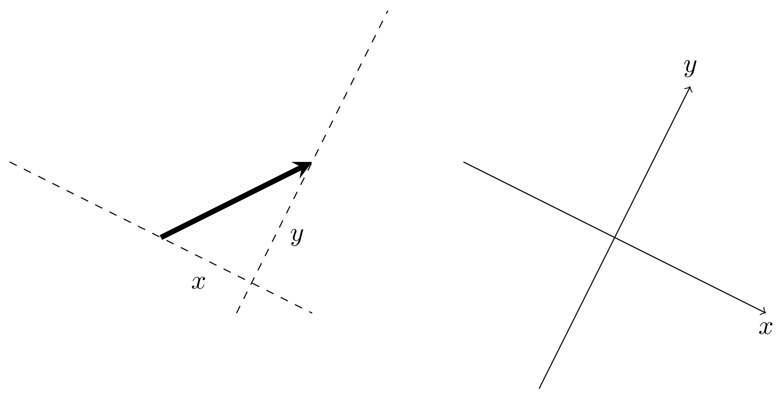


Figure 8: The steps to find the Cartesian coordinates of a vector. The dashed lines are drawn parallel to the axes and form a right triangle with the vector. If you know the length of the vector and can find one of the angles, then you can use trigonometry to find the lengths of the two other sides of the triangle. These give you the x and y coordinates of the vector, although they don't tell you whether the coordinates are positive or negative. To figure this out, just look at whether the vector is going in the same direction (+ sign) or opposite direction (- sign) to the axis. In this case, both coordinates are positive because the vector is closer to pointing along the positive x and y directions.

Notating Cartesian Coordinates Vectors are usually written in bold ($\mathbf{v}$) or with an arrow on top ($\vec{v}$). The magnitude (length) of a vector is written as $|\vec{v}|$ or just v (without an arrow on top). The *components* of a vectors can be written in different ways. Here is an example of the coordinates of the same vector written in different notations:

- $v_x = 2$, $v_y = -3$, $v_z = 1$
- $\vec{v} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$
- $\vec{v} = 2\hat{x} - 3\hat{y} + \hat{z}$

- $\vec{v} = 2\hat{i} - 3\hat{j} + \hat{k}$

Working Backwards from Cartesian Coordinates The magnitude (length) of a vector is given by the Pythagorean Theorem:

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2}.$$

1.4.3 Adding and Subtracting Vectors

There are two ways to add and subtract vectors. One is visual, and the other is to add or subtract the Cartesian coordinates.

Visual Method The visual method for adding and subtracting vectors is illustrated in Figure 9.

To visually add vectors, put the tail of the second vector at the end of the first vector. Then the sum is the vector that starts at the tail of the first vector and ends at the end of the second vector.

To visually subtract vectors, we use a trick. Suppose we were asked to find $\vec{v}_1 - \vec{v}_2$. This is the same as $\vec{v}_1 + (-\vec{v}_2)$. Now $-\vec{v}_2$ looks like the $\vec{v}_2$ vector, except it is flipped to point in the opposite direction. All we need to do to visually subtract is to first find $-\vec{v}_2$ by flipping $\vec{v}_2$, and then to add $\vec{v}_1$ and $-\vec{v}_2$ using the method above.

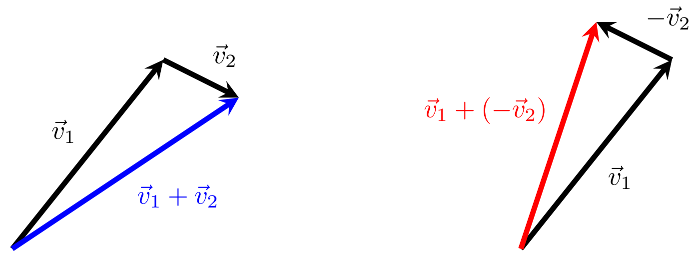


Figure 9: The visual method for adding and subtracting vectors.

Coordinates Method The other way (which is the method we will be using almost always in this course) to add or subtract vectors is to add or subtract the coordinates. To do this, find the x, y, and z coordinates of each vector. Then add the x coordinates together to get the x coordinate of the sum. Add the y coordinates together to get the y coordinate of the sum, and do the same for the z component if needed.

Remember: Never add or subtract the magnitudes of vectors directly. 5 miles north + 2 miles south is not 7 miles (in any direction). You need to find the individual coordinates and add/subtract those.

1.4.4 Trigonometry Review

Here are some useful facts to remember from trigonometry. Make sure you know all of this. You will need this information!

- For a right triangle, $\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$, $\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$, and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$.
- **Pythagorean Theorem**: If c is the length hypotenuse (longest side) of a right triangle, and a and b are the lengths of the other two sides, then $c^2 = a^2 + b^2$.
- Given two parallel lines, and a third line that intersects them, the alternate interior angles are equal (see Figure 10).
- If a 180° angle is divided up into smaller angles, the sum of all those angles must be 180°. Same for 90° (see Figure 11).
- The angles of a triangle add to 180° (see Figure 12).
- The two acute angles of a right triangle add to 90° (see Figure 13).

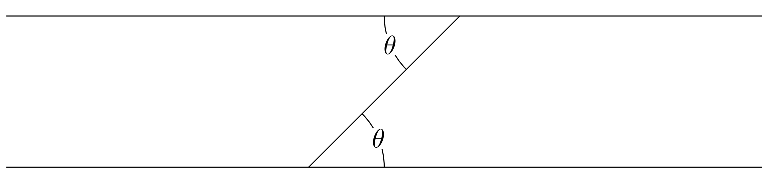


Figure 10: Given two parallel lines, and a third line that intersects them, the alternate interior angles are equal.

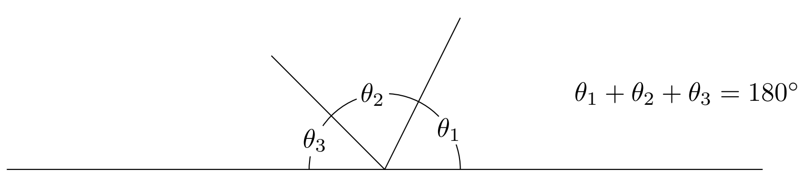


Figure 11: If a 180° angle is divided up into smaller angles, the sum of all those angles must be 180°. Same for 90°.

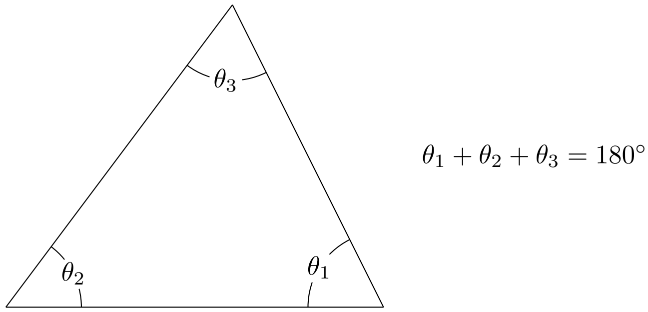


Figure 12: The angles of a triangle add to 180°.

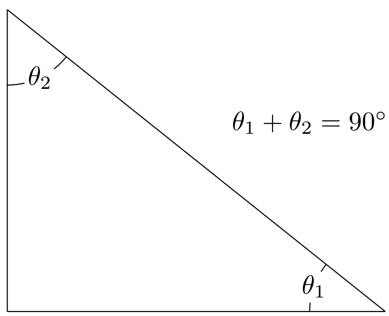


Figure 13: The two acute angles of a right triangle add to 90°.

2 Vectors and Kinematics

In this course, we will frequently need to discuss the position, velocity, and acceleration of physical objects.

2.1 Calculus Review

In calculus, you should have learned about derivatives and integrals. The derivative of a function $f(t)$ with respect to the variable t is the rate at which f changes when t changes. If $f(t)$ is graphed as a function of t, the derivative of f with respect to t is the slope of the graph. If $f(t)$ were a line, its slope would be given by the formula $\frac{f(t+\Delta t)-f(t)}{\Delta t}$. If $f(t)$ is more complicated, however, this formula only gives the *average* slope:

$$\text{average slope of } f \text{ with respect to } t = \frac{f(t+\Delta t)-f(t)}{\Delta t}.$$

The derivative $\frac{df}{dt}$ is defined as the instantaneous slope. To get the instantaneous slope, we need to take the limit as $\Delta t \to 0$:

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t}.$$

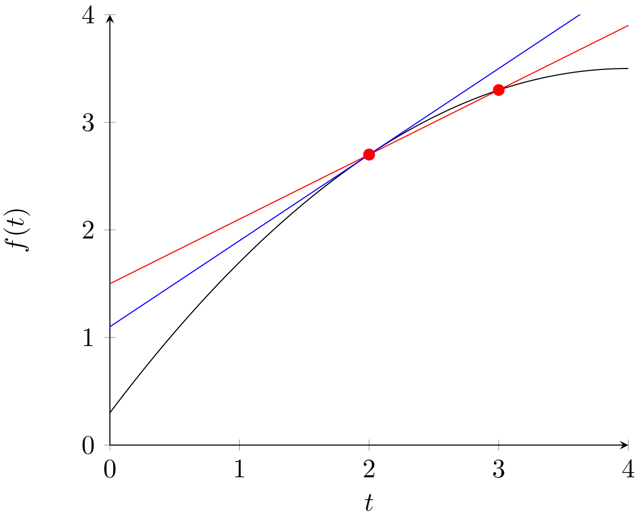


Figure 14: The derivative of $f(t)$ with respect to t is the slope of the graph of f versus t. The blue line show the instantaneous slope at the point $t = 2$, and the red line shows the average slope between $t = 2$ and $t = 3$. The average slope is different from the instantaneous slope, but would become closer and closer as the red points are moved closer together.

The integral of a function is, intuitively, the "area under the curve" (the area under the graph of a function). Definite integrals (which are generally what we will be using in this course), have a specific starting and ending point. For example, $\int_{t_i}^{t_f} f(t)dt$ is the integral of f with respect to t, with t_i and the starting point (i stands for "initial") and t_f is the ending point (f stands for "final").

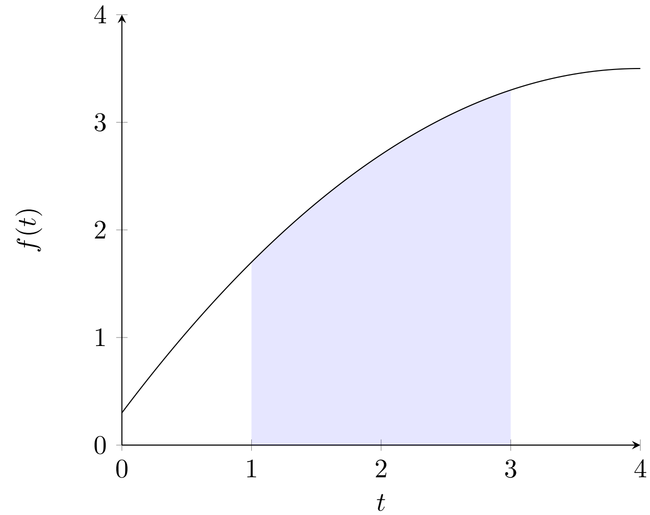


Figure 15: The integral $\int_1^3 f(t)dt$ is the blue-shaded area under the graph of f versus t between times 1 and 3. Note that the phrase "area under the curve" is a little ambiguous, since it doesn't specify the bottom boundary. In the case of a definite integral, the correct bottom boundary is the bottom axis. Anything below that axis would be counted as negative area.

One important result from your calculus classes is the Fundamental Theorem of Calculus, which says

$$f(t_f) - f(t_i) = \int_{t_i}^{t_f} \frac{df}{dt} dt.$$

In other words, if you integrate a derivative, you get back the original function (or at least the difference $f(t_f) - f(t_i)$ between the original function at two points).

Because we will often have expressions like $f(t_f) - f(t_i)$, we define the delta notation Δf to mean

$$\Delta f \equiv f(t_f) - f(t_i).$$

Note that this is "final minus initial" (the order is important). We could also define $\Delta t = t_f - t_i$. Using this delta notation, the average rate of change formula above becomes:

$$\text{average slope of } f \text{ with respect to } t = \frac{\Delta f}{\Delta t}.$$

2.2 Position

We can use vectors to represent position. However, a vector by itself is not enough. Position is relative! If I send a text saying, "the library is 1 mile north," that is not enough information. I also need to communicate the starting location. The library is not 1 mile north from New York City, for example. To properly specify location, I would need to say, "1 mile north from my house," for example. In this case, "1 mile north" is a vector, and "my house" is the reference location.

In physics, we often call the reference location "the origin." When drawing axes, we often draw them so that they intersect at the origin. All locations are then given by vectors. The vector tells you how far (the magnitude) and in what direction you need to go to get from the origin to the desired location.

2.3 Brief Note on Time

Time is also relative. Whenever we give times in physics, we need to specify what our "zero" time is. For example, I might say that time $t = 0$ corresponds to the moment when a ball is dropped. Then I could describe the motion of the ball at different points in time after it is dropped. For example, $t = 2$s would be two seconds after the ball is dropped.

2.4 Velocity

Velocity is defined as the rate of change of position with respect to time. Velocity is a vector. The magnitude of the velocity is the speed of the object, and the direction of the velocity is the direction in which the object is moving.

In calculus, you learned how to calculate rates of change. If I give you a position vector that is a function of time, like

$$\vec{x}(t) = \begin{bmatrix} 6 \\ 15 + 3t - 10t^2 \end{bmatrix},$$

the velocity vector (the rate of change of position) can be found by taking the derivative of each coordinate of the position vector:

$$\vec{v}(t) = \vec{x}'(t) \equiv \frac{d\vec{x}}{dt} = \begin{bmatrix} \frac{d}{dt}6 \\ \frac{d}{dt}(15 + 3t - 10t^2) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 - 20t \end{bmatrix}.$$

We can also use calculus to work backwards and go from velocity to (the change in) position:

$$\Delta\vec{x} \equiv \vec{x}(t_f) - \vec{x}(t_i) = \int_{t_i}^{t_f} \frac{d\vec{x}}{dt} dt = \int_{t_i}^{t_f} \vec{v}(t) dt.$$

2.5 Acceleration

Acceleration, which is a vector just like velocity, is defined as the rate of change of velocity with respect to time. When an object accelerates, its velocity changes. This can mean either that the magnitude of the velocity changes (speeding up or slowing down), or that the direction of the velocity changes. For example, if a car goes at a constant speed around a curve, it is accelerating: the direction of its velocity is changing.

We can once again use calculus to calculate acceleration based on velocity. The (time) derivative of velocity is acceleration:

$$\vec{a}(t) = \frac{d\vec{v}}{dt}.$$

We can also go in the opposite direction using integration, just like before:

$$\Delta\vec{v} = \vec{v}(t_f) - \vec{v}(t_i) = \int_{t_i}^{t_f} \frac{d\vec{v}}{dt} dt = \int_{t_i}^{t_f} \vec{a}(t) dt.$$

Variable	Derivative	Integral
$\vec{x}$	$\vec{v} = \frac{d\vec{x}}{dt}$	-
$\vec{v}$	$\vec{a} = \frac{d\vec{v}}{dt}$	$\Delta x = \int_{t_i}^{t_f} \vec{v} dt$
$\vec{a}$	-	$\Delta v = \int_{t_i}^{t_f} \vec{a} dt$

Figure 16: Summary of kinematic variables and their derivatives/integrals. $\vec{x}$ is position, $\vec{v}$ is velocity, and $\vec{a}$ is acceleration. $\Delta\vec{x}$ means "the change in $\vec{x}$," and is equal to $\vec{x}(t_f) - \vec{x}(t_i)$.

3 Constant Acceleration

3.1 The Kinematic Equations

Now we'll use the information from Section **??** to find equations for $\vec{x}$ and $\vec{v}$ for the specific case of an object that has constant acceleration. **All the formulas in this section will only apply to objects that have constant acceleration.**

Imagine we have an object that, for some reason, is accelerating at a constant rate. The acceleration would be some vector $\vec{a}$ that does not depend on time (it is constant). If we want to find the velocity, we can use the formula from Figure **??**:

$$\Delta\vec{v}(t) = \int_{t_i}^{t_f} \vec{a} dt$$

Since $\vec{a}$ is a constant, we can factor it out of the integral:

$$\Delta\vec{v}(t) = \int_{t_i}^{t_f} \vec{a} dt = \vec{a} \int_{t_i}^{t_f} dt = (t_f - t_i)\vec{a}$$

$$\implies \vec{v}(t_f) - \vec{v}(t_i) = (t_f - t_i)\vec{a}$$
$$\implies \vec{v}(t_f) = \vec{v}(t_i) + (t_f - t_i)\vec{a}.$$

Sometime, you may see $\vec{v}(t_i)$ written as $\vec{v}_0$. In many problems, we also choose $t_i = 0$ and just write t_f as t. In that case, this formula becomes

$$\vec{v}(t) = \vec{v}_0 + t\vec{a}.$$

Now that we have found $\vec{v}(t)$, we can also find $\Delta\vec{x}$ using the formula from Figure **??**:

$$\Delta\vec{x}(t) = \int_{t_i}^{t_f} \vec{v}(t)dt = \int_{t_i}^{t_f} [\vec{v}_0 + t\vec{a}]\, dt = (t_f - t_i)\vec{v}_0 + \frac{1}{2}(t_f^2 - t_i^2)\vec{a}$$

$$\implies \vec{x}(t_f) - \vec{x}(t_i) = (t_f - t_i)\vec{v}_0 + \frac{1}{2}(t_f^2 - t_i^2)\vec{a}$$

$$\implies \vec{x}(t_f) = \vec{x}(t_i) + (t_f - t_i)\vec{v}_0 + \frac{1}{2}(t_f^2 - t_i^2)\vec{a}.$$

Again, we will often write $\vec{x}(t_i)$ as $\vec{x}_0$, choose $t_i = 0$, and relabel t_f as just t. Then the formula becomes

$$\vec{x}(t) = \vec{x}_0 + t\vec{v}_0 + \frac{1}{2}t^2\vec{a}.$$

3.2 Graphs of Constant-Acceleration Problems

In Section 3.1, we used calculus to find the formulas that describe the velocity and position of an object in constant acceleration. We can also think about what the graphs of each kinematic variable (acceleration, velocity, and position) look like for an object in constant acceleration. These graphs are portrayed in figures 17, 18, and 19.

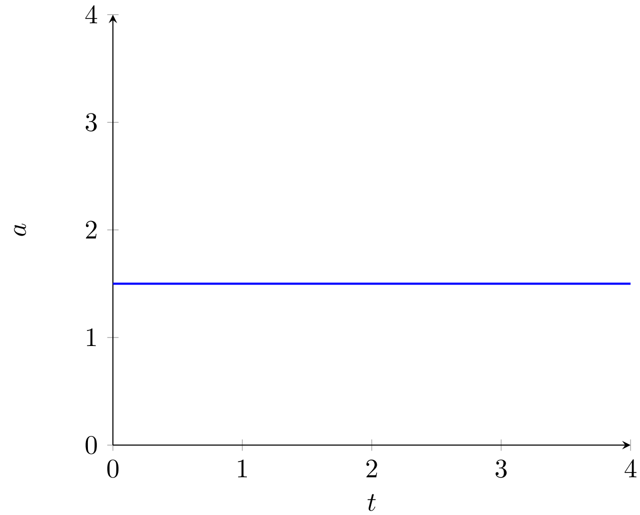


Figure 17: For constant acceleration, the graph of acceleration versus time will just be a flat line: the acceleration stays at the same value for all time.

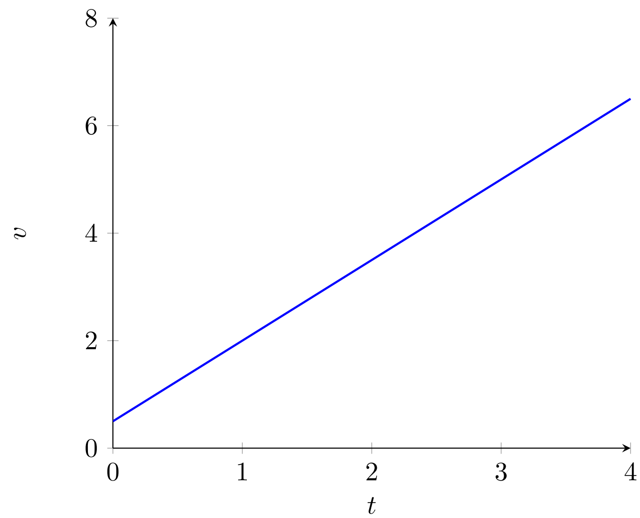


Figure 18: For constant acceleration, the graph of velocity versus time will just be a straight line. The slope of the line is the acceleration, and the slope must be constant because the acceleration is constant. We can see the initial value of the velocity by looking at the value of the graph at time $t = 0$.

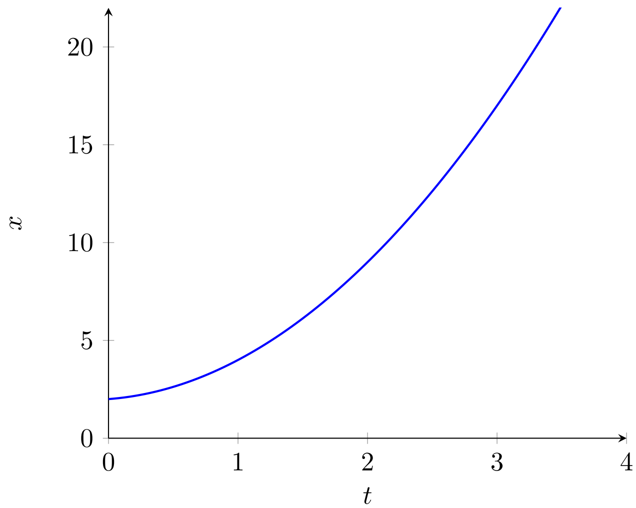


Figure 19: For constant acceleration, the graph of position versus time will be a parabolic curve. The graph can curve either upwards or downwards, depending on the acceleration and initial velocity.

3.3 Gravity

Gravity causes objects to fall with a constant acceleration. The acceleration due to gravity is given by

$$g \approx 9.81 \text{ m/s}^2.$$

Note that this gives the *magnitude* of the gravitational acceleration. The direction is always pointing straight down toward the center of the earth (unless you are in outer space). In a one-dimensional problem, you should make sure the sign of g matches your choice of axis. If you have defined the positive direction as being upward, then g should be negative, because gravity pulls downward.

One useful piece of information when dealing with problems involving gravity is that, if an object is thrown straight up, its velocity when it reaches the highest point in its path is zero. This is because the object was initially traveling upward, but is about to turn around and start traveling downward. Because the object cannot instantly change from an upward velocity to a downward velocity, it must, for a brief moment, have zero velocity.

3.4 Working with Kinematic Equations

The kinematic equation we have found so far (which only apply to problems with constant acceleration) are

$$\vec{v}(t) = \vec{v}_0 + t\vec{a}.$$

$$\vec{x}(t) = \vec{x}_0 + t\vec{v}_0 + \frac{1}{2}t^2\vec{a}.$$

In one-dimension, we can drop the vector symbols and just write numbers for x, v, and a (although always remember the the sign of the number determines its direction). Then the equations are just

$$v(t) = v_0 + ta.$$

$$x(t) = x_0 + tv_0 + \frac{1}{2}t^2 a.$$

These equations can be recombined using algebra to get an additional useful equation. First, we can solve the v equation for t:

$$v(t) = v_0 + ta \implies t = \frac{v - v_0}{a}.$$

Then we can substitute this equation for t into the equation for x:

$$x(t) = x_0 + tv_0 + \frac{1}{2}t^2 a = x_0 + \left(\frac{v - v_0}{a}\right) v_0 + \frac{1}{2}\left(\frac{v - v_0}{a}\right)^2 a$$

$$\implies v^2 = v_0^2 + 2a(x(t) - x_0) \equiv v_0^2 + 2a\Delta x.$$

This equation might be useful for certain problems (again, it only applies to problems with constant acceleration). The kinematic equations we have derived in this section are summarized in Figure 20.

$a(t) = \text{constant}$
$v(t) = v_0 + ta$
$x(t) = x_0 + tv_0 + \frac{1}{2}t^2 a$
$v^2 = v_0^2 + 2a\Delta x$

Figure 20: The kinematic equations for problems with constant acceleration.

4 Projectile Motion

In this class, we will discuss how to apply the kinematic equations for constant acceleration (Figure **??**) to calculate the path of a projectile. Once a projectile is dropped or thrown through the air, gravity will make the object accelerate downward at a constant rate of $g \approx 9.81$ m/s^2. Because the acceleration is constant, we can apply the kinematic equation from Figure **??**. However, we need to be careful. Projectile motion usually happens in two dimensions (unless the object is falling straight up or down). This means that we need to account for both the magnitude and direction.

The kinematic equation from Figure ?? apply to each coordinate separately. This is an important point! You should never try to use the kinematic equations with the magnitude of a vector (unless you are dealing with a one-dimensional problem). Always break the vector up into x and y coordinates first. See Figure 21.

$a_x(t) = \text{constant}$	$a_y(t) = \text{constant}$
$v_x(t) = v_{0,x} + ta_x$	$v_y(t) = v_{0,y} + ta_y$
$x(t) = x_0 + tv_{0,x} + \frac{1}{2}t^2 a_x$	$y(t) = y_0 + tv_{0,y} + \frac{1}{2}t^2 a_y$
$v_x^2 = v_{0,x}^2 + 2a_x\Delta x$	$v_y^2 = v_{0,y}^2 + 2a_y\Delta y$

Figure 21: The kinematic equations in component form.

Let's start with acceleration. Gravity gives a constant acceleration of $g \approx 9.81\ \text{m/s}^2$ downward.

Activity: Choose a coordinate system, and find the x and y coordinates of the gravitational acceleration.

Solution: See Figure 22.

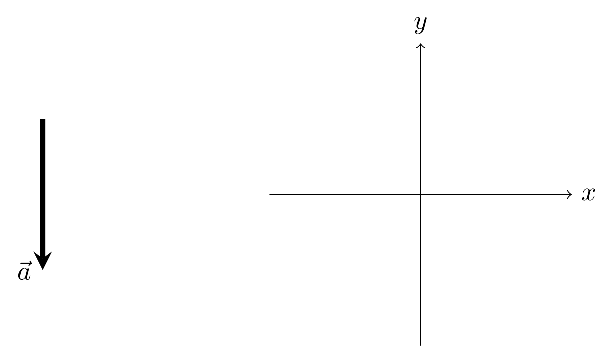
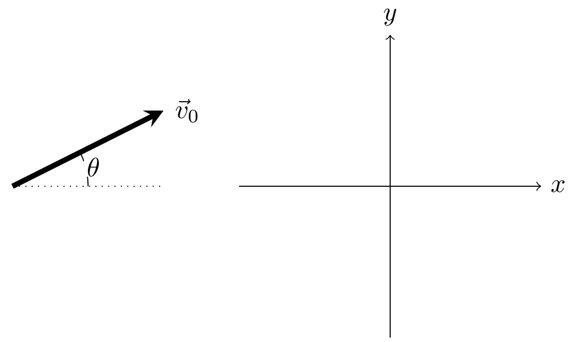


Figure 22: The gravitational acceleration points straight down along the y-axis, so $a_x = 0$ and $a_y \approx -9.81\text{m/s}^2$.

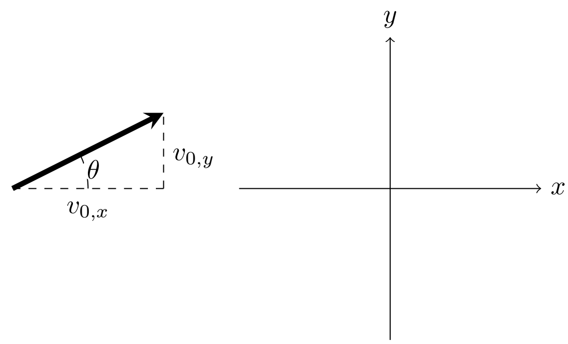
Once we know $a_x = 0$ and $a_y \approx -9.81\ \text{m/s}^2$, we can move on to velocity. In the kinematic equations for velocity, we need to know v_0. Sometimes, the initial velocity is given in the problem. For example, a problem might say that a projectile is launched with some initial speed $|\vec{v}_0|$ at an angle of θ above the horizontal. This gives us the initial velocity vector, since it gives both magnitude and direction.

Activity: Choose a coordinate system, and find the x and y coordinates of the initial velocity vector.

Solution: See Figure 23.



(a) The initial velocity has magnitude $|\vec{v}_0|$ and a direction that is at angle θ above the horizontal.



(b) We draw lines parallel to the axes to make a right triangle. Then we can use trigonometry to find the coordinates: $\sin\theta = v_{0,y}/|\vec{v}_0|$ and $\cos\theta = v_{0,x}/|\vec{v}_0|$. Solving these equations gives $v_{0,x} = |\vec{v}_0|\cos\theta$ and $v_{0,y} = |\vec{v}_0|\sin\theta$

Figure 23: Solution to initial velocity activity.

Once we've found $v_{0,x} = |\vec{v}_0|\cos\theta$ and $v_{0,y} = |\vec{v}_0|\sin\theta$, we can use the kinematic equations for $\vec{v}$ to get

$$v_x(t) = v_{0,x} + ta_x = |\vec{v}_0|\cos\theta + t\cdot 0 = |\vec{v}_0|\cos\theta$$

$$v_y(t) = v_{0,y} + ta_y = |\vec{v}_0|\sin\theta + t\cdot(-9.81)\ \mathrm{m/s}^2.$$

From the first equation, notice that v_x is constant (it doesn't change with time). In fact, it has the same value as $v_{0,x}$. The reason v_x is constant is that gravity only causes acceleration in the downward direction (which lies only along our y-axis). Because all the acceleration is in the y-direction, the x-coordinate of the velocity does not change. On the other hand, v_y *is* affected by gravity and decreases with time.

So far, we have found the equations for $\vec{a}$ and $\vec{v}$. We can also find the equation for $\vec{x}$. Following a common convention for position, we will use the variable x (without an arrow on top) to represent the x-component of $\vec{x}$, and we will use y to represent the y-component. Based on the kinematic equations in Figure **??**,

$$x = x_0 + tv_{0,x} + \frac{1}{2}t^2 a_x = x_0 + t|\vec{v}_0|\cos\theta$$

$$y = y_0 + tv_{0,y} + \frac{1}{2}t^2 a_y = y_0 + t|\vec{v}_0|\sin\theta + \frac{(-9.81 \text{ m/s}^2)}{2}t^2.$$

$a_x = 0$	$a_y = -g$
$v_{0,x} = \|\vec{v}_0\|\cos\theta$	$v_{0,g} = \|\vec{v}_0\|\sin\theta$
$v_x(t) = \|\vec{v}_0\|\cos\theta$	$v_y(t) = \|\vec{v}_0\|\sin\theta - tg$
$x = x_0 + t\|\vec{v}_0\|\cos\theta$	$y = y_0 + t\|\vec{v}_0\|\sin\theta - \frac{g}{2}t^2.$

Figure 24: The equations of motion for a projectile. θ is the angle of the initial velocity **above the horizontal direction**.

4.1 Projectile Range

The range of a projectile is the distance the projectile travels before it lands. We can use the equations of motion we found to predict the range of a projectile. We will assume the projectile starts at ground-level. If we define $y = 0$ to be ground-level as well, then this means $y_0 = 0$. The projectile lands once $y(t) = 0$. The y equation is

$$y(t) = y_0 + t|\vec{v}_0|\sin\theta - \frac{g}{2}t^2.$$

Putting in $y_0 = 0$ and $y = 0$ gives

$$0 = t|\vec{v}_0|\sin\theta - \frac{g}{2}t^2$$

$$\implies 0 = |\vec{v}_0|\sin\theta - \frac{g}{2}t$$

$$\implies t = \frac{2|\vec{v}_0|\sin\theta}{g}$$

In this equation, t is the landing time. By solving this equation, we were able to find the time at which the projectile will land. To find the range, we just need to figure out how far the projectile traveled in the x direction. The x equation is

$$x(t) = x_0 + t|\vec{v}_0|\cos\theta.$$

If we choose $x = 0$ to be the location where the projectile started, then $x_0 = 0$. If we also use $t = \frac{2|\vec{v}_0|\sin\theta}{g}$, we get

$$x = \left(\frac{2|\vec{v}_0|\sin\theta}{g}\right)|\vec{v}_0|\cos\theta = \frac{2|\vec{v}_0|^2\sin\theta\cos\theta}{g}.$$

Using the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$, we can simplify this to

$$\boxed{R = \frac{|\vec{v}_0|^2\sin 2\theta}{g}}.$$

I have replaced x with R to emphasize that this is the range of the projectile. This equation is sometimes called the range equation. It is only valid if is the starting height is at ground level (in other words, if $y_0 = 0$).

4.2 Throwing versus Dropping

Suppose I drop an object straight down. Its velocity and height are given by

$$v_y(t) = v_{0,y} - (9.81\ \mathrm{m/s}^2)t.$$

$$y(t) = y_0 + tv_{0,y} - \frac{(9.81\ \mathrm{m/s}^2)}{2}t^2.$$

If I drop it, then $v_{0,y} = 0$ (the object initially is not moving). If I throw it straight down, it might have some non-zero initial velocity $v_{0,y}$.

Now consider a different situation where I throw an object through the air. Now it will move in both the x and y directions. However, the equations for the y-velocity and height are still

$$v_y(t) = v_{0,y} - (9.81\ \mathrm{m/s}^2)t.$$

$$y(t) = y_0 + tv_{0,y} - \frac{(9.81\ \mathrm{m/s}^2)}{2}t^2.$$

The fact that the object is also moving in the x direction does not affect its motion in the y direction.

One thing to remember in particular is, just like we discussed in Lecture 2.1, when an object moving under gravity reaches its maximum height, its upward velocity is zero. For a projectile, when the projectile reaches the highest point of its flight, its vertical velocity is zero (although its horizontal velocity is not necessarily zero).

5 Relative Motion and Circular Motion

5.1 Relative Motion

Worksheet: Suppose I am traveling in a train moving at 30 m/s. Suppose I walk toward the front of the train at a speed of 1 m/s (relative to the floor of the train). How fast am I moving relative to the ground?

Solution: Because the train is already moving, and because I am moving in the same direction as the train is moving, my total speed relative to the ground is 30 m/s + 1 m/s = 31 m/s.

Worksheet: Suppose instead I walk toward the back of the train. How fast am I moving relative to the ground?

Solution: Because I am now moving in the opposite direction to the train, my total speed relative to the ground is 30 m/s - 1 m/s = 29 m/s.

Worksheet: Suppose instead I walk across the width of the train, from one side to the other. How fast am I moving relative to the ground?

Solution: The previous discussion questions were all one-dimensional. This problem is two-dimensional, because the train and I are moving in different directions. See Figure 25 for a drawing of the situation and a choice of axes. With these axes, $v_{\text{me},x} = 0$ and $v_{\text{me},y} = -1$ m/s, and $v_{\text{me},x} = 30$ m/s and $v_{\text{me},y} = 0$. Therefore, if I sum my velocity relative to the train with the train's velocity, I would get $v_x = 30$ m/s and $v_y = -1$ m/s. Using the Pythagorean theorem to get the total magnitude, $|v| = \sqrt{v_x^2 + v_y^2} = \sqrt{901} \approx 30.02$ m/s.

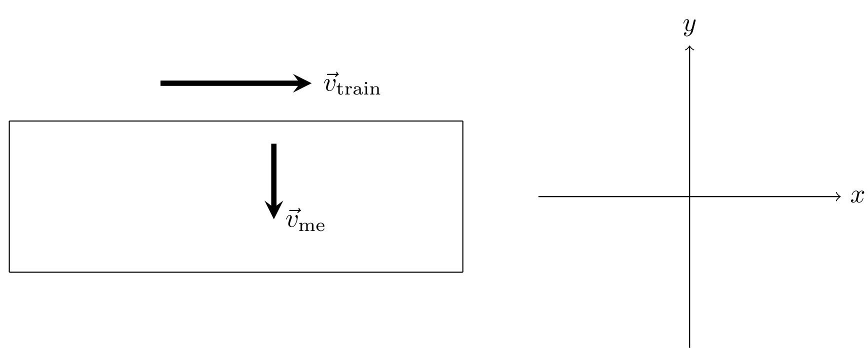


Figure 25: Diagram of the train problem.

5.1.1 Notation for Relative Motion

In the previous examples, the train was one reference frame, and the ground was another reference frame. We could measure velocities relative to the train, or relative to the ground. We will sometimes use subscripts to represent different reference frames. For example, the subscript G could stand for "ground," the subscript T could stand for "train," and the subscript M could stand for "me". Then we could write the velocity of the train relative to the ground as $\vec{v}_{TG}$. The velocity of me relative to the train would be $\vec{v}_{MT}$, and the velocity of me relative to the ground would be $\vec{v}_{MG}$. In this notation, the equation to find my velocity relative to the ground would be

$$\vec{v}_{MG} = \vec{v}_{MT} + \vec{v}_{TG}.$$

More generally, if we have an object A and reference frames labeled by B and C, then this equation would be

$$\boxed{\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}.}$$

5.1.2 Airplane Example

Problem: A plane is moving east at 200 m/s relative to the air. It is moving 250 m/s relative to the ground. The air is moving south. What is the speed of the air relative to the ground?

Solution: See Figure 26. We know $|\vec{v}_{PG}|$ (the speed of the plane relative to the ground) is 250 m/s, and that $|\vec{v}_{PA}|$ (the speed of the plane relative to the air) is 200 m/s east. Since $\vec{v}_{PA}$ is pointed east, using the coordinate system from Figure 26, we can conclude that $\vec{v}_{PA,x} = 200$ m/s and $\vec{v}_{PA,y} = 0$. We don't know $\vec{v}_{AG}$, but we do know that it is pointed south, so $\vec{v}_{AG,x} = 0$ and $\vec{v}_{AG,y}$ is some negative number, which we want to find. We can use the relative velocity equation

$$\vec{v}_{PG} = \vec{v}_{PA} + \vec{v}_{AG}.$$

Remember that we need to add vectors using coordinates (not magnitudes). Looking at the coordinates above, we can see that

$$\vec{v}_{PG,x} = \vec{v}_{PA,x} + \vec{v}_{AG,x} = 200 \text{ m/s}$$

and

$$\vec{v}_{PG,y} = \vec{v}_{PA,y} + \vec{v}_{AG,y} = \vec{v}_{AG,y}.$$

Now we can use our knowledge that $|\vec{v}_{PG}| = 250$ m/s. By the Pythagorean theorem,

$$(250 \text{ m/s})^2 = (200 \text{ m/s})^2 + \vec{v}^2_{AG,y}$$

$$\implies \vec{v}^2_{AG,y} = (250 \text{ m/s})^2 - (200 \text{ m/s})^2 = 102500 \text{ m}^2/\text{s}^2$$

$$\implies \vec{v}_{AG,y} = \pm\sqrt{102500} \text{ m/s} \approx 320 \text{ m/s}.$$

We were asked to find the speed of the air relative to the ground:

$$|\vec{v}_{AG}| = \sqrt{\vec{v}^2_{AG,x} + \vec{v}^2_{AG,y}} = \sqrt{0 + 102500 \text{ m}^2/\text{s}^2} \approx 320 \text{ m/s}.$$

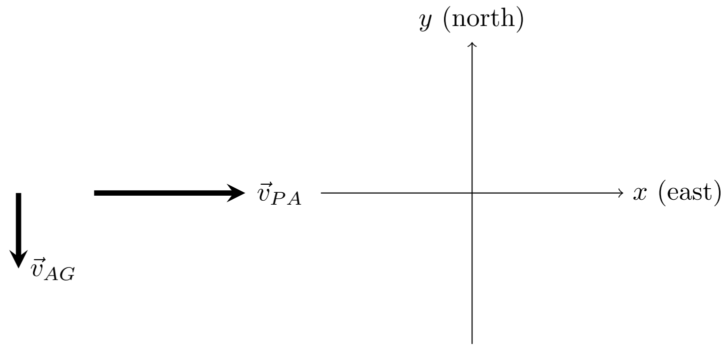


Figure 26: A diagram for the plane activity. $\vec{v}_{PG}$ is the velocity of the plane relative to the ground, and $\vec{v}_{AG}$ is the velocity of the air relative to the ground

5.2 Circular Motion

5.2.1 Velocity and Acceleration

Question: See Figure 27.

Solution: When the string breaks, the rock will travel in a straight line tangent to the circle (path B in Figure 27).

Remember: When an object is moving in circular motion, its velocity will always be tangent to the circle.

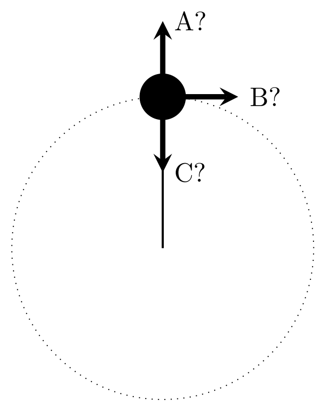
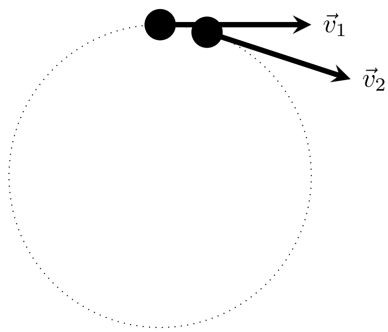
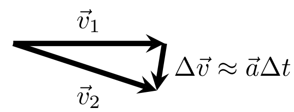


Figure 27: Suppose a rock at the end of a string is being swung in a circle. Which way will the rock travel if the string suddenly breaks?

An object in circular motion is always accelerating. This is because the direction of its velocity vector is always changing. The acceleration of an object in circular motion points toward the inside of the circle, as illustrated in Figure 28.



(a) The velocity of an object in circular motion at two points in time.



(b) The difference between the two velocities $\Delta\vec{v}$ is approximately proportional to the acceleration.

Figure 28: When an object moves in a circle, the acceleration vector is pointed toward the inside of the circle. In uniform circular motion, the acceleration points exactly toward the center of the circle.

5.2.2 Uniform Circular Motion

When an object moves in a circle at a constant speed, we call this motion *uniform circular motion.* Note that even though the *speed* is constant in uniform circular motion, the *velocity* is not constant because the object's direction is constantly changing.

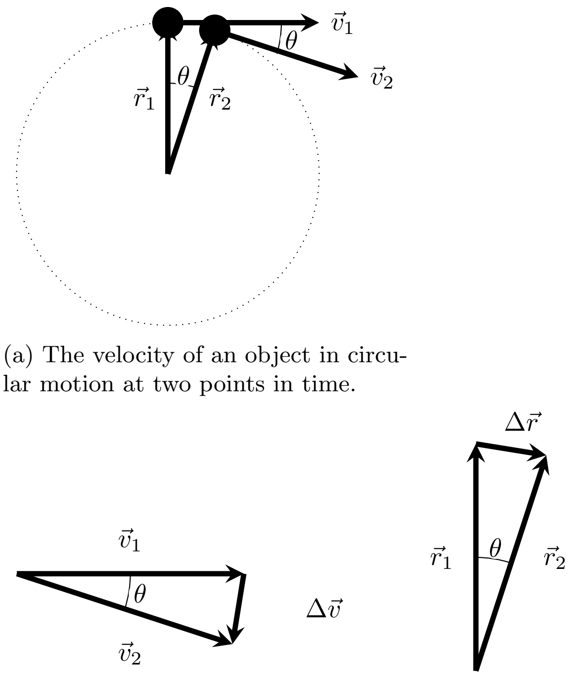
The acceleration of an object in uniform circular motion is always pointed exactly toward the center of the circle. The reason this is true is because if the acceleration were not perpendicular to the velocity, the magnitude of the velocity vector (the speed) would increase. Uniform circular motion means the object is traveling at a constant speed, and so the acceleration is perpendicular to the velocity. Because the velocity is always tangent to the circle, the direction perpendicular to the velocity is toward the center of the circle.

The magnitude of the acceleration (of an object in uniform circular motion) is given by the equation

$$\boxed{|\vec{a}_c| = \frac{|\vec{v}|^2}{r},}$$

where a_c is the magnitude of the acceleration, $|\vec{v}|$ is the magnitude of the velocity, and r is the radius of the circle. This equation is derived in Figure 29.

Remember: The acceleration of an object in uniform circular motion is always pointed directly toward the center of the circle and has magnitude $\frac{|\vec{v}|^2}{r}$.



(a) The velocity of an object in circular motion at two points in time.

(b) The velocity triangle.

(c) The radius triangle.

Figure 29: The "velocity triangle" and "radius triangle" are similar triangles. The angle θ is the same for both because $\vec{v}_1$ is perpendicular to $\vec{r}_1$ and $\vec{v}_2$ is perpendicular to $\vec{r}_2$. The single angle θ is enough to ensure the triangles are similar because the triangles are isosceles (remember $\vec{v}_1$ and $\vec{v}_2$ have the same length because speed is constant in uniform circular motion). Because the triangles are similar, $\frac{|\Delta\vec{v}|}{|\vec{v}_1|} = \frac{|\Delta\vec{r}|}{|\vec{r}_1|}$. We can divide both sides by Δt and rearrange to get $\frac{|\Delta\vec{v}|}{\Delta t} = \frac{|\vec{v}_1|}{|\vec{r}_1|}\frac{|\Delta\vec{r}|}{\Delta t}$. Now we can take the limit as $\Delta t \to 0$ to get $\frac{d|\vec{v}|}{dt} = \frac{|\vec{v}_1|}{|\vec{r}_1|}\frac{d|\vec{r}|}{dt}$, or using the definitions of $\vec{a}$ and $\vec{v}_1$, $|\vec{a}| = \frac{|\vec{v}_1|^2}{|\vec{r}_1|}$.

6 Newton's Laws of Motion

Newton's three laws of motion are

1. A moving object keeps moving at a constant velocity unless acted upon by a force.

2. The acceleration of an object is proportional to the sum of all forces acting on it:
$$m\vec{a} = \sum_i \vec{F}_i$$

(the notation $\sum_i \vec{F}_i \equiv \vec{F}_1+\vec{F}_2+\cdots$ just means "sum all the force vectors").

3. If object A causes a force on object B, then object B causes a force on object A with equal magnitude but opposite direction (this is sometimes stated as "for every action, there is an equal but opposite reaction").

Newton's second law implies the first law. If there are no forces, Newton's second law says that the acceleration will be zero, and therefore the velocity will not change.

6.0.1 Newton's First Law

If I were on a train which was moving in a straight line at a constant speed on a perfectly smooth track (no bumps or vibrations), then the only way I could know that I am moving is by looking out the window. There is no physics experiment that I could do within the train to figure out that I am moving.

In the context of mechanics, Newton's first law is the reason that I cannot figure out that I am moving. For example, as we discussed in Lecture 2.2, if I drop a ball, the ball will have a horizontal velocity equal to the horizontal velocity of the train. However, since I am also moving horizontally at the exact same speed, I will not be able to see that the ball is moving horizontally. To me, it will look like the ball is falling straight down, just as if I were not moving.

6.0.2 Newton's Second Law

Newton's second will be very important in this course. The three variables involved are mass m, acceleration $\vec{a}$, and the forces $\vec{F}_i$. We have already discussed acceleration in previous lectures. Now we will discuss mass and forces.

Mass

We can solve the equation in Newton's second law for mass:

$$m = \frac{|\sum_i \vec{F}_i|}{|\vec{a}|}.$$

Based on this, we can see that mass is "force per acceleration" (the amount of force required to get a given amount of acceleration).

Mass measures how hard it is to move an object. If an object has a lot of mass, then a lot of force is required to accelerate the object. The standard units of mass in the SI system are kilograms (note that the standard unit is kilograms, not grams). Note that mass is not the same as weight. Weight measures how heavy an object is (or in other words, how much gravity pulls on the object). On the other hand, mass measures how hard it is to move an object. The weight of an object can be different on different planets (because each planet has a different gravitational strength), but the mass of an object stays the same the same even when no gravity is acting on it at all. There is a connection between weight and mass which we will discuss below in the section on gravity.

Discussion: If we plot the magnitude of the total force versus the magnitude of acceleration, what will the graph look like?

Solution: By Newton's second law,

$$|\sum_i \vec{F}_i| = m|\vec{a}|.$$

If we replace the magnitude of the total force $|\sum_i \vec{F}_i|$ with y, and the magnitude of the acceleration $|\vec{a}|$ with x, then this equation would be

$$y = mx.$$

This is just the equation for a line with a slope equal to the mass m.

6.0.3 Force

Force is a vector. A force acts on an object by pulling it in some direction. Whenever we use Newton's second law, we always need to sum over all the forces acting on an object. Once we know the total force $\sum_i \vec{F}_i$, we can put it into the equation for Newton's second law.

Because force is a vector, we need to break all the forces up into x and y coordinates before adding them together. The Newton's second law equation applies to the x and y coordinates separately:

$$ma_x = \sum_i F_{i,x}$$
$$ma_y = \sum_i F_{i,y}.$$

Note that the mass m is not a vector and does not need to be broken up into x and y components.

The units of force can be determined based on Newton's second law. The units of $m\vec{a}$ are kilograms times meters per second squared (written $\mathrm{kg} \cdot \mathrm{m/s}^2$). Since $m\vec{a}$ is equal to a sum of forces, the units for force must be the same. For convenience, the SI system defines a derived unit called a "Newton" (abbreviated "N") that means $\mathrm{kg} \cdot \mathrm{m/s}^2$. For example, 5 N means $5\ \mathrm{kg} \cdot \mathrm{m/s}^2$.

Activity: A force pulls an object of mass 5 kilograms downward with a strength of 4 N, and another force pulls the same object toward the right with a force of 3 N. What is the magnitude of the total acceleration of the object?

Solution: Let's call the forces $\vec{F}_1$ and $\vec{F}_2$. If we use a standard set of axes, then $F_{1,x} = 0$, $F_{1,y} = -4$ N, $F_{2,x} = 3$ N, and $F_{2,y} = 0$. The total force in the x direction is $F_{1,x} + F_{x,2} = 0 + 3\ \mathrm{N} = 3\ \mathrm{N}$. Therefore, the x-coordinate of acceleration is

$$a_x = \frac{\sum_i F_{i,x}}{m} = \frac{3\ \mathrm{N}}{5\ \mathrm{kg}} = \frac{3\ \mathrm{kg} \cdot \mathrm{m/s}^2}{5\ \mathrm{kg}} = \frac{3}{5}\ \mathrm{m/s}^2.$$

Note that we had to use the definition of the unit N in order to simply the expression to the correct units for acceleration. The total force in the y direction is $F_{1,y} + F_{y,2} = -4\text{ N} + 0 = -4\text{ N}$, and so

$$a_y = \frac{\sum_i F_{i,y}}{m} = -\frac{4}{5}\text{ m/s}^2.$$

The magnitude of the acceleration is given by the Pythagorean theorem,

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2} = \sqrt{\frac{9}{25} + \frac{16}{25}}\text{ m/s}^2 = \sqrt{\frac{25}{25}}\text{ m/s}^2 = 1\text{ m/s}^2.$$

6.0.4 Newton's Third Law

Newton's third law states that whenever an object pushes on another object, the other object pushes back. For example, if I push on the wall with a force of 2 N, then the wall will push back on my hand with a force of 2 N in the opposite direction.

6.1 Working with Forces

6.1.1 Free-body Diagrams

When working with forces, it is often helpful to write a free-body diagram. A free-body diagram is a picture of an object in the problem with all of the force vectors drawn on it. The force vectors are usually drawn starting from the center of the object. Drawing a free-body diagram can help you keep track of all the forces in the problem and which directions they are acting in.

6.1.2 Equilibrium

An object is said to be in *equilibrium* when the sum of all the forces acting on an object is zero (mathematically, this means $\sum_i \vec{F}_i = 0$). In this situation, all the forces cancel each other out. By Newton's second law, an object in equilibrium will not acceleration ($\vec{a} = 0$). This means it will keep moving at a constant velocity.

6.2 Different Types of Forces

6.2.1 The Gravitational Force

The gravitational force always pulls downward toward the ground (at least when we are near the surface of the earth). The magnitude of the gravitational force acting on an object of mass m is

$$\boxed{|\vec{F}_g| = mg.}$$

Remember that the gravitational acceleration $g = 9.81 \text{ m/s}^2$. Note that the gravitational *force* $\vec{F}_g$ is not the same as the gravitational *acceleration* g (don't get mixed up and write $F_g = g$).

We can use Newton's second law to find the acceleration due to gravity. If gravity is the only force acting on the object, then

$$m|\vec{a}| = |\vec{F}_g| = mg$$

$$\implies |\vec{a}| = \frac{mg}{m} = g.$$

Notice that the acceleration due to gravity does not depend on the mass m of the object. This is why I said earlier that gravity gives all objects a constant acceleration of magnitude g.

Notice that the magnitude of the gravitational force on an object is proportional to its mass. This is the connection between weight and mass. On the surface of the earth, weight (the force of gravity on an object) is proportional to mass. The proportionality constant is g (unless we are in outer space or on a different planet).

6.2.2 Tension

The force on an object when it is being pulled by a string is called tension. The tension force always pulls along the direction of the string. When two objects are connected by a string, the magnitude of the string tension is the same at both ends of a string, as long as there's nothing in between. For example, if a weight is hanging from the ceiling by a string, the magnitude of the tension on the weight is the same as the magnitude of the tension on the ceiling. On the other hand, if a weight is hanging from the ceiling and a second weight is tied to the string in between the ceiling and the first weight, then the magnitude of the tension on the bottom weight is not the same as the magnitude of the tension on the ceiling.

6.2.3 Normal Forces

When objects come into contact with each other, they exert forces on each other. These forces (usually) prevent objects from passing through each other. The direction of the normal force is perpendicular to the surface of the object that is causing the force. The magnitude of the normal force depends on the exact situation.

Activity: An object of mass m is resting on a table. The only forces acting on the object are gravity $\vec{F}_g$ and the normal force $\vec{F}_N$ from the table. The object is not moving. Find the magnitude of the normal force.

Solution: Because the object is not moving, we know that it is in equilibrium. Therefore, the sum of all forces must be zero: $\vec{F}_g + \vec{F}_N = 0$. The direction of the gravitational force is downward, so $F_{g,x} = 0$ and $F_{g,y} = -mg$. The direction of the normal force is perpendicular to the table, which means it is pointing upward. Therefore, $F_{N,x} = 0$. We don't know what $F_{N,y}$ is, but we know that $F_{g,x} + F_{N,y} = 0 \implies F_{N,y} = -F_{g,y} = mg$. The magnitude of the normal force is therefore $|\vec{F}_N| = mg$.

7 Newton's Laws Examples

7.1 Rotated Coordinates

A ship is moving at a constant speed at an angle of 25° south of the westward direction. The wind is blowing and causing a force of 2000 N on the ship pushing it south. The ship's rudder causes a force perpendicular to the ship's velocity. The drag force from the water is pointed directly against the ship's velocity. Finally, the ship's propellor causes a force of $8,000$ N that pushes the ship forward.

Activity: Draw axes and a free-body diagram for the ship.

Solution: See Figure 30.

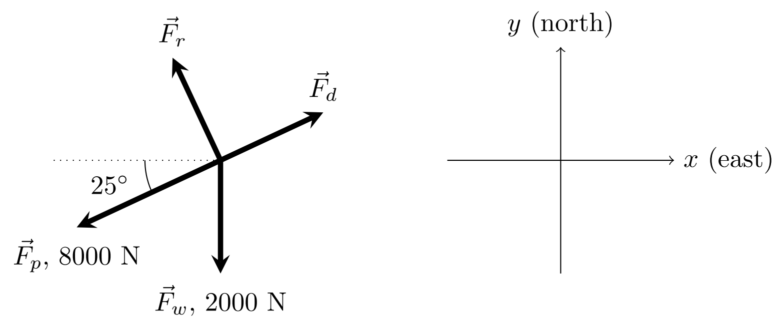


Figure 30: Free-body diagram of the ship problem.

Activity: With the axes in Figure 30, $\vec{F}_w$ points along the y-axis, but no other vectors lie along an axis. Choose a rotated coordinate system so that three vectors lies along axes, and only one does not lie along an axis.

Solution: See Figure 31.

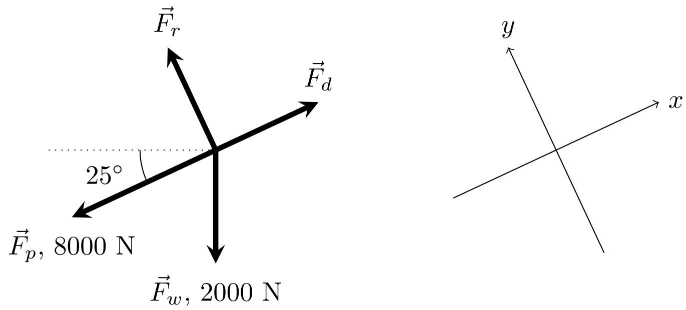


Figure 31: Free-body diagram of the ship problem with a rotated coordinate system. Now $\vec{F}_p$, $\vec{F}_r$, and $\vec{F}_d$ lie along the axes. Only $\vec{F}_w$ does not lie along an axis.

Activity: Write down Newton's second law for the ship.
Solution: Newton's second law is

$$\sum_i \vec{F}_i = m\vec{a}.$$

This is a two-dimensional problem, so we need to break this equation up into x and y components:

$$\sum_i F_{i,x} = ma_x$$
$$\sum_i F_{i,y} = ma_y.$$

We are told in the problem statement that the ship is traveling at a constant speed. Therefore, the acceleration of the ship is 0 (whenever you see the words "constant speed," this is usually important information). When we put $a_x = 0$ and $a_y = 0$ into Newton's second law, we get

$$\sum_i F_{i,x} = 0$$

$$\sum_i F_{i,y} = 0.$$

If we write out the sum explicitly, remembering to include all the forces, we get

$$F_{p,x} + F_{r,x} + F_{d,x} + F_{w,x} = 0$$

$$F_{p,y} + F_{r,y} + F_{d,y} + F_{w,y} = 0.$$

Most of these components are easy to find because of our choice of coordinate system.

- $F_{p,x} = -8000$ N, because $\vec{F}_p$ lies along the negative x-axis and has a magnitude of 8000 N.

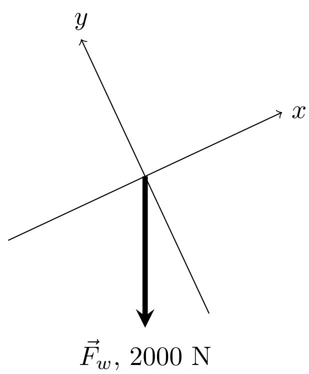
- $F_{r,x} = 0$ because $\vec{F}_r$ lies along the y-axis, not the x-axis.
- $F_{d,x}$ is unknown.
- $F_{p,y} = 0$ because $\vec{F}_p$ lies along the x-axis.
- $F_{r,y}$ is unknown.
- $F_{d,y} = 0$ because $\vec{F}_d$ lies along the x-axis.

The difficult force is $\vec{F}_w$. Because $\vec{F}_w$ does not lie along one of the axes, we need to do more work to find its x and y components. When we work out the geometry (see Figure 32), we find that $F_{w,x} = -|\vec{F}_w| \sin 25° = (-2000\ \text{N}) \sin 25° \approx -845$ N (negative because it lies closer to the negative x direction) and $F_{w,y} = (-2000\ \text{N}) \cos 25° \approx -1813$ N. If we put all of this into the equations we got from Newton's second law, we get

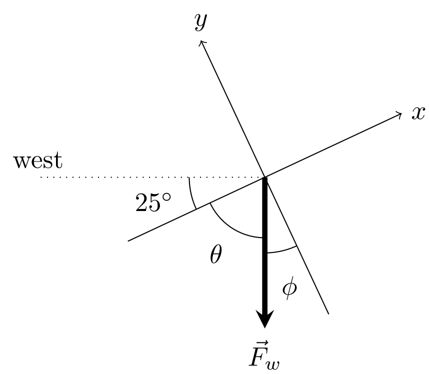
$$F_{p,x} + F_{r,x} + F_{d,x} + F_{w,x} = -8000\ \text{N} + 0 + F_{d,x} - 845\ \text{N} = 0$$
$$\implies F_{d,x} = 8845\ \text{N}.$$

and

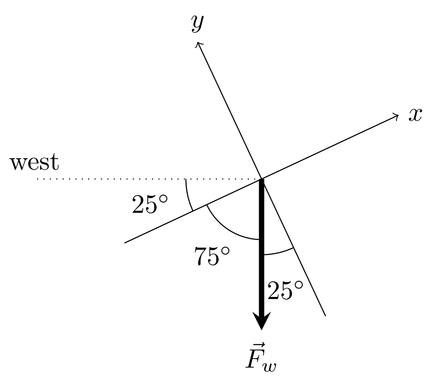
$$F_{p,y} + F_{r,y} + F_{d,y} + F_{w,y} = 0 + F_{r,y} + 0 - 1813\ \text{N} = 0$$
$$\implies F_{r,y} = 1813\ \text{N}.$$



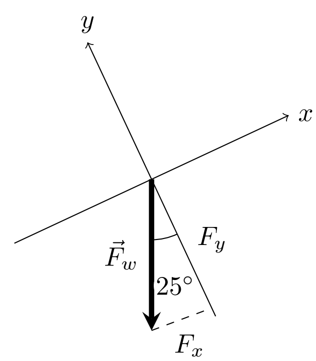
(a) Start by drawing a set of axes through the tail of $\vec{F}_w$.



(b) We know the velocity is 25° below westward direction. The x-axis in our figure lies along the same direction as the velocity. Therefore, the x-axis is 25° below the westward direction. We can draw this on the figure.



(c) We can now find θ and ϕ using the fact that $25° + \theta = 90°$ and $\theta + \phi = 90°$.



(d) Draw a right triangle with the vector as the hypotenuse and sides parallel to the axes. In this case, I chose the y axis as one side and the dashed line as the other side.

Figure 32: Based on the triangle $\sin 25° = \frac{F_x}{|\vec{F}_w|}$ and $\cos 25° = \frac{F_y}{|\vec{F}_w|}$.

7.2 Ramps

A block sits on a perfectly smooth ramp (no friction). The ramp is at an angle of 50° above the ground.

Activity: Draw a free-body diagram with "normal" (horizontal and vertical) axes. If the block starts to slide down the ramp, will there be acceleration in the x direction? Will there be acceleration in the y direction

Solutions: See Figure 33. If gravity starts to pull the block down the ramp, the combination of gravity and the normal force will cause acceleration in both the x and y direction.

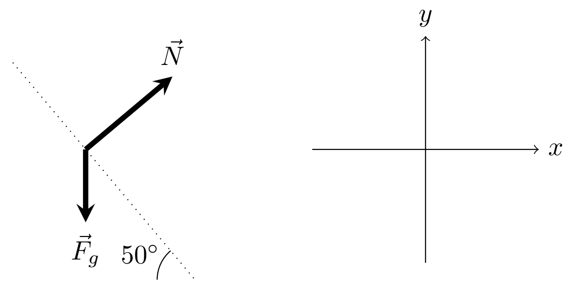


Figure 33: Free-body diagram of the ramp problem with "normal" axes.

Activity: Draw a free-body diagram with rotated axes (so that the x-axis lies along the direction of the ramp). Will there be acceleration in the x direction? Will there be acceleration in the y direction

Solutions: See Figure 34. Since the block will stay on the ramp as it moves, it will only move along the x axis, and it will never move along the y axis. Therefore, the x acceleration will be non-zero, but the y acceleration will be zero.

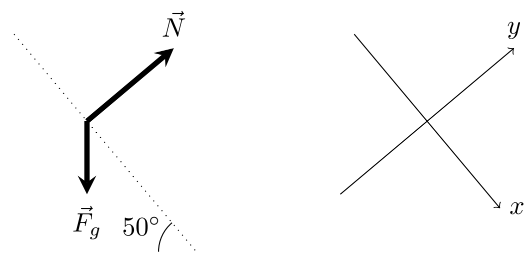


Figure 34: Free-body diagram of the ramp problem with rotated axes.

Activity: If the block has a mass of 5 kg, what is the acceleration of the block?

Solutions: We can use Newton's second law to find acceleration. As discussed in the last section, Newton's second law in component form is

$$N_x + F_{g,x} = ma_x$$

$$N_y + F_{g,y} = ma_y.$$

Note that we are using $\vec{N}$ for the normal force (this could also be written as $\vec{F}_N$). We know that

- $m = 5$ kg.

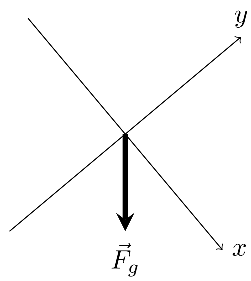
- $N_x = 0$, because the normal force is only along the y direction (perpendicular to the ramp).
- N_y is unknown.
- $|F_g| = mg = 5 \text{ kg} \cdot 9.8 \text{ m/s}^2 = 49.0 \text{ kg} \cdot \text{ m/s}^2 = 49.0 \text{ N}$.

We want to find a_x. The only missing piece of information is $F_{g,x}$. To find this, we need to break up $\vec{F}_g$ into components. The result of the geometry in Figure 35 is that $F_{g,x} = |\vec{F}_g| \cos 40° = 49 \text{ N} \cos 40° \approx 25.7 \text{ N}$. If we put all the values we found into the Newton's second law equation, we get

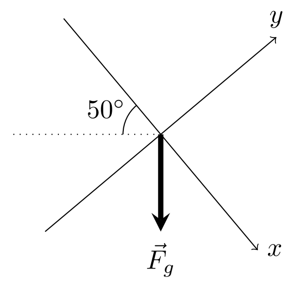
$$N_x + F_{g,x} \approx 0 + 25.7 \text{ N} = (5 \text{ kg})a_x$$

$$\implies a_x \approx \frac{25.7 \text{ N}}{5 \text{ kg}} \approx 5.14 \frac{\text{kg} \cdot \text{m/s}^2}{\text{kg}} = 5.13 \text{ m/s}^2.$$

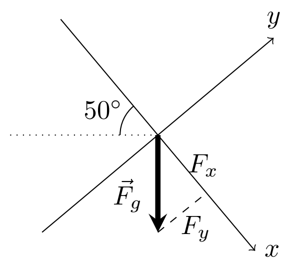
As we discussed before, $a_y = 0$, so the total acceleration is just 5.13 m/s^2.



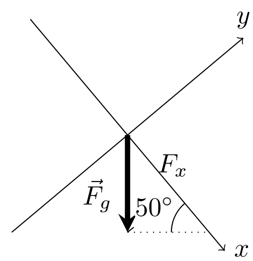
(a) Start by drawing a set of axes through the tail of $\vec{F}_g$.



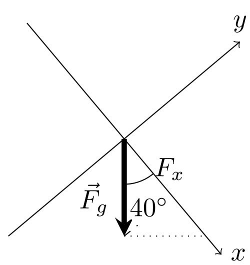
(b) We know that the x-axis makes an angle of $50°$ with the horizontal.



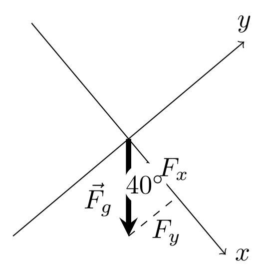
(c) Draw a right triangle with the vector as the hypotenuse and sides parallel to the axes. In this case, I chose the x axis as one side and the dashed line as the other side.



(d) If we draw any horizontal line, it will still make an angle of $50°$ with the x-axis. In particular, we can draw a horizontal line to form (another) right triangle with the vector.



(e) This lets us find the other angle.



(f) Now we know one of the angles of the original right triangle.

Figure 35: Based on the triangle $\sin 40° = \frac{F_y}{|\vec{F}_g|}$ and $\cos 40° = \frac{F_x}{|\vec{F}_g|}$.

7.3 Multiple Hanging Masses

A rock of mass M_1 is hanging by a rope. Another rock of mass M_2 is hanging by a rope tied to the first rock (see Figure 36).

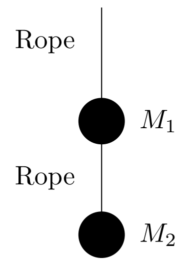


Figure 36: Two hanging masses.

Activity: Draw a free-body diagram for M_1 and a free-body diagram for M_2.

Solution: See Figure 37.

Activity: Find the tension on the top string $|\vec{T}_1|$ and the tension on the bottom string $|\vec{T}_2|$.

Solution: We will again use Newton's second law. This time, however, there are two objects. Newton's second law applies to both of them. This is a one-dimensional problem (all forces lie along the up-down direction), and so we don't need to break up Newton's second law into components. Therefore, Newton's second law gives us two equations (one for each mass):

$$\sum_i F_{i,M_1} = M_1 a_1$$

$$\sum_i F_{i,M_2} = M_2 a_2.$$

$\sum_i F_{i,M_1}$ is the sum of all the forces acting on the first mass, and $\sum_i F_{i,M_2}$ is the sum of all the forces acting on the second mass. If we write out the sum explicitly, referring to the free-body diagram, we get

$$T_1 - T_2 - F_{g,M_1} = M_1 a_1$$

$$T_2 - F_{g,M_2} = M_2 a_2.$$

We know that

- $a_1 = a_2 = 0$ (both masses are not moving).
- $|F_{g,M_1}| = M_1 g$.
- $|F_{g,M_2}| = M_2 g$.

Putting in this information, Newton's second law becomes

$$T_1 - T_2 - gM_1 = 0$$

$$T_2 - gM_2 = 0.$$

We can solve the second equation for T_2 to get $T_2 = gM_2$, and then we can replace T_2 in the first equation with gM_2 to get

$$T_1 - gM_2 - gM_1 = 0$$

$$\implies T_1 = g(M_1 + M_2).$$

So the tension in the bottom string is just $T_2 = gM_2$, and the tension in the top string is $T_1 = g(M_1 + M_2)$. This makes a lot of sense if you think about it. The second string only need to hold up the weight of the second mass, but the first string need to hold up the weight of the first mass and the second mass.

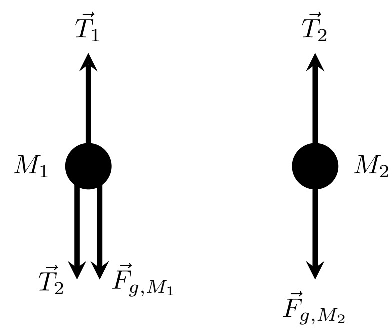


Figure 37: The first mass has the tension of the top string $\vec{T}_1$ pulling up, gravity pulling down, and the tension of the second string $\vec{T}_2$ pulling down. The second mass has the tension of the second string $\vec{T}_2$ pulling up and gravity pulling down.

7.4 Atwood's Machine

Atwood's machine is made of a pulley and two hanging masses. The masses are connected to each other by a string and hang from the pulley, with one mass on each side of the pulley (see Figure 38).

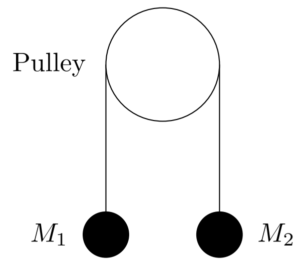


Figure 38: One version of Atwood's machine. Two masses connected to the same string are hanging from a pulley.

Activity: Draw a free-body diagram for each mass in Atwood's machine.
Solution: See Figure 39.

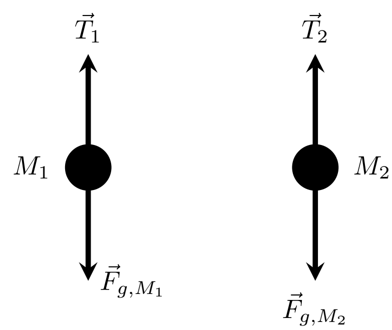


Figure 39: The free-body diagram for Atwood's machine.

Activity: Find the acceleration of each mass.
Solution: This is a one-dimensional problem. Even though the masses are in two different locations, they can each only move up and down, and all forces are in the up-down direction. Newton's second law for the first mass and for the second mass gives:

$$T_1 - M_1 g = M_1 a_1$$

$$T_2 - M_2 g = M_2 a_2.$$

There are two important things we can figure out about Atwood's machine:

- $T_1 = T_2$. Because both masses are connected by the same rope, and because there is nothing in between the masses except the pulley (which we will assume turns freely without any effort), both tensions have the same magnitude (and direction, because they both point up).

- $a_1 = -a_2$. Because both masses are connected by a rope, when one of them speeds up in one direction, the other will also speed up in the opposite direction.

Given this information, we will just write T instead of T_1 and T_2. If we put this information into Newton's second law, we get

$$T - M_1 g = M_1 a_1$$

$$T - M_2 g = -M_2 a_1.$$

We can solve the second equation to get $T = M_2 g - M_2 a$. Putting this into the first equation, we get

$$M_2 g - M_2 a_1 - M_1 g = M_1 a_1$$

$$\implies a_1 = g\frac{(M_2 - M_1)}{(M_1 + M_2)}.$$

Then since $a_2 = -a_1$,

$$\implies a_2 = g\frac{(M_1 - M_2)}{(M_1 + M_2)}.$$

8 Circular Motion, Hooke's Law, and Friction

8.1 Circular Motion

A car is traveling along a circular path of radius 250 m at a constant speed of 30 m/s. As it is driving, it goes over some ice. The ice reduces the friction between the car and the road to zero. Normally, the car would slide off the road.

We can choose the y axis to lie along the direction that the car is moving when it hits the ice, and the x axis to point toward the center of the circle (see Figure 40).

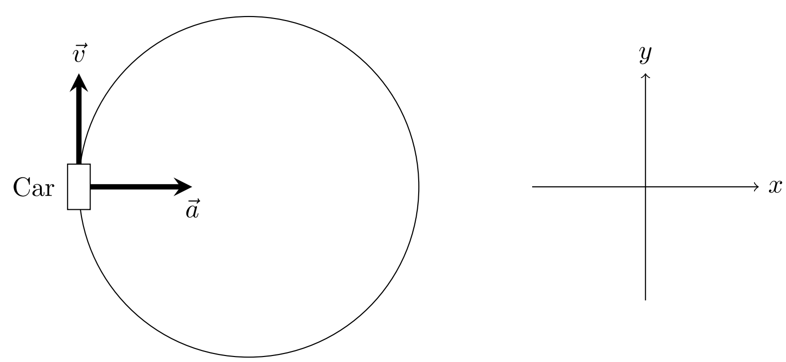


Figure 40: A car traveling in a circle.

Remember that, since the car is traveling on a circular path at a constant speed, the car is in uniform circular motion. In uniform circular motion, the car has an acceleration of magnitude $\frac{v^2}{R}$ and a direction pointing towards the center of the circle. According to Newton's second law, the amount of total forces required to produce this acceleration is

$$\sum_i F_{i,x} = ma_x = m\frac{v^2}{R}.$$

Normally this force would be provided by friction. However, without friction, the only forces on the car are gravity and the normal force, which point along the up-down direction (in our picture, this would be into and out of the page).

Discussion: How can the road be designed so that the car will not slide off? Assume that the only forces acting on the car will be the normal force and gravity.

Solution: We cannot change the direction of the gravitational force, but we can change the direction of the normal force by tilting the road (see Figure 41).

Figure 41: We can tilt the road so that the normal force has an x-component pointing along the x-direction toward the center of the circle.

Activity: What is the angle θ required so that the car will continue moving along the circle?

Solution: I'm going to change my y axis to point in the upward direction (instead of in the direction of the car's velocity as in Figure 40). By Newton's second law:

$$N_x + 0 = ma_x = m\frac{v^2}{R} \implies N_x = \frac{mv^2}{R}$$
$$N_y - mg = ma_y = 0 \implies N_y = mg.$$

Based on Figure 42,

$$\phi = \arctan\left(\frac{N_y}{N_x}\right) = \arctan\left(\frac{mg}{mv^2/R}\right) = \arctan\left(\frac{gR}{v^2}\right).$$

Using $v = 30$ m/s, $R = 250$ m, and $g = 9.8$ m/s^2, we get

$$\phi \approx 69.8°,$$

which means

$$\theta \approx 20.17°.$$

Notice that we did not need to use the mass m of the car, so this angle would work for cars of any mass.

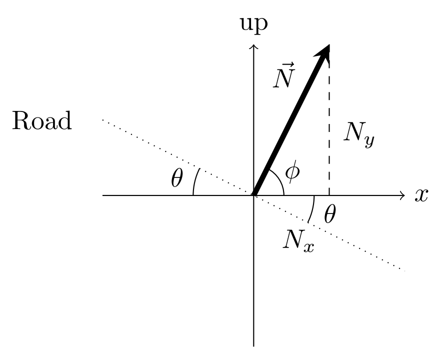


Figure 42: The between the road (the dotted line) and the horizontal (the x-axis) is θ. We can use the fact that opposite angles are equal to see that θ is also the angle of the dotted line below the x-axis. Finally, we can use the fact that $\theta + \phi = 90°$ to get $\phi = 90° - \theta$. We can use the fact that $\phi = \arctan(\frac{\text{opposite}}{\text{adjacent}}) = \arctan(\frac{N_y}{N_x})$ to find ϕ.

8.2 Hook's Law

When a spring or other elastic material is stretched or compressed, it exerts a force that tries to return the spring to its natural length. The change in the length of the spring is often written as $\Delta x = x_f - x_0$, where x_0 is the natural length of the spring, and x_f is its length after being compressed or stretched.

Discussion: A spring of unknown length is stretched by 0.02 m. What is Δx?

Solution: Even though we don't know the original length of the spring, we know Δx is the change in length, and so $\Delta x = \pm 0.02$ m (the sign depends on which direction is positive).

Discussion: If Δx is positive, what should the force of the spring $\vec{F_s}$ be positive or negative?

Solution: If $\Delta x = x_f - x_0$ is positive, that means $x_f > x_0$. In other words, the final position of the spring lies further in the positive direction than the natural length of the spring. Since the spring wants to return to its natural position, the force of the spring will pull in the negative direction.

Every spring is different. However, we can use approximations for the force caused by a spring. The simplest approximation would be $F_{\text{spring}} = k$, where k is some constant that depends on the spring.

Discussion: Does the approximation $F_{\text{spring}} = k$ make sense?

Solution: The approximation $F_{\text{spring}} = k$ does not make sense. It would imply that the force of a spring pushes in the same direction (positive if k is positive, or negative if k is negative) regardless of whether you stretch the spring or compress.

We expect that F_{spring} should depend on Δx in some way (the force of a spring depends on how much you change its length). We could make the guess that

$$F_{\text{spring}} = k_1\Delta x + k_2\Delta x^2 + k_3\Delta x^3 + \cdots ,$$

where each k_i is some constant that might depend on which spring we are using. This form is very general. Many functions can be written, or at least approximated, by choosing different values for each k_i. However, this form is also complicated and not very useful. However, if we make the approximation that Δx is small, then the terms $k_2\Delta x^2 + k_3\Delta x^3 + \cdots$ will be even smaller than the term $k_1\Delta x$. For example, if $\Delta x = 0.1$, then $\Delta x^2 = 0.01$ which is even smaller, and $\Delta x^3 = 0.001$ which is even smaller. As an approximation, we can ignore all the terms with higher powers of Δx and just use

$$\boxed{F_{\text{spring}} = -k\Delta x.}$$

I included a minus sign to remind us that the force has the opposite sign compared to Δx. This is called Hooke's law. It is often a good approximation for the force of a spring. The constant k depends on the properties of the spring, and is called the *spring constant*.

8.3 Friction

Another force that can vary a lot depending on the circumstances is friction. Friction is the name for forces which resist the motion of an object.

Discussion: An object is moving in the negative x direction. What is the direction of friction?

Solution: Friction resists motion. Since the object is moving in the negative x direction, friction will point in the positive x direction.

Discussion: An object is at rest. Is it being pulled in the positive x direction with a force of 2 N. The only other force acting on the object is friction. What is the magnitude and direction of the force of friction F_f?

Solution: From Newton's second law, and the fact that the object is not moving, we know that the total force in the x direction must be 0. Since the only forces are friction and the force of 2 N, these force must cancel out. Since the force of 2 N is pulling in the positive x direction, friction must be pulling with an equal force of 2 N in the negative x direction (resisting the motion).

Even though friction is complicated, we can use simple approximations. To make these approximations, we need to distinguish between two types of friction. *Static friction* is the force of friction when an object is at rest (not moving). *Kinetic friction* is the force of friction when an object is moving. Static friction is usually stronger than kinetic friction. This is because of various processes at a molecular level. The force of static friction always has whatever magnitude and direction it needs to prevent an object from moving. The maximum amount of force from static friction is

$$\boxed{|F_s| \leq \mu_s|\vec{N}|,}$$

where μ_s (the coefficient of static friction) is some constant that depends of the object and the surface on which it is resting, and $|\vec{N}|$ is the magnitude of the normal force between the object and the surface. Once the static friction reaches this magnitude, it cannot go any higher, and so the object will start moving. Once the object starts moving, the magnitude of the force of kinetic friction is

$$\boxed{|F_k| = \mu_k|\vec{N}|,}$$

where μ_k is the coefficient of kinetic friction, which once again depends on the surface and the object.

8.3.1 Example

Activity: An box of mass $m = 25$ kg is being pulled across a flat table by a spring. The box starts moving only when the spring is stretched by 1 m. The coefficient of static friction is $\mu_s = 0.6$. Find the spring constant k.

Solution:
The horizontal direction:
Right before the box starts moving, the static friction must be at its maximum level $F_s = \mu_s|\vec{N}|$. Therefore, at this moment, the spring must be providing a force of equal magnitude $|F_{\text{spring}}| = |F_s| = \mu_s|\vec{N}|$.

The vertical direction:
The force of gravity and the normal force must add to zero. Therefore, $|N| = |F_g| = mg$.

We found $|F_{\text{spring}}| = \mu_s|N| = \mu_s mg$. On the other hand, by Hooke's law, $|F_{\text{spring}}| = k\Delta x$. Therefore,

$$k\Delta x = \mu_s mg$$

$$\implies k = \frac{\mu_s mg}{\Delta x} = \frac{0.6 \cdot 25\text{ kg} \cdot 9.8\text{ m/s}^2}{1\text{ m}} = 147\text{ N/m}.$$

9 The Work-Energy Theorem

9.1 The Dot Product

We have talked about how to add vectors, either visually or using Cartesian coordinates, but we have not talked about how to multiply vectors. The dot product is one way to multiply two vectors together to get a single number. The dot product of $\vec{A}$ and $\vec{B}$ is defined as

$$\boxed{\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta,}$$

where θ is the angle between $\vec{A}$ and $\vec{B}$. Notice that

- $|\vec{A}||\vec{B}| \cos\theta$ is a number, not a vector. It has a magnitude, but it does not have a direction.
- If the vectors are perpendicular ($\theta = 90°$), then $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos 90° = 0$.
- If the vectors are pointing in the same direction ($\theta = 0$), then $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos 0° = |\vec{A}||\vec{B}|$.
- If the vectors are pointing in the opposite direction ($\theta = 180°$), then $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos 180° = -|\vec{A}||\vec{B}|$.

There is another equation we can use to calculate the dot product:

$$\boxed{\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y.}$$

This equation gives exactly the same result as the previous equation. You can use either equation to calculate the dot product. Depending on what information you already have, one equation might be easier to use than the other.

9.1.1 Constant Dot Product

If the dot product $\vec{F}_{\text{tot}} \cdot d\vec{r} = |F_{\text{tot}}| \cos\theta dr$ is constant, then the work is just

$$\boxed{W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{\text{tot}} \cdot d\vec{r} = |F_{\text{tot}}| \cos\theta \int_{\text{path}} dr = |F_{\text{tot}}||\Delta r| \cos\theta.,}$$

where $|\Delta\vec{r}|$ is the total length of the path along which the object travels from $\vec{r}_i$ to $\vec{r}_f$.

9.2 The Work-Energy Theorem

Newton's second law says that

$$m\vec{a} = \sum_i \vec{F}_i.$$

If we define $\vec{F}_{\text{tot}} = \sum_i \vec{F}_i$ and remember that $\vec{a} = \frac{d\vec{v}}{dt}$, this equation becomes

$$m\frac{d\vec{v}}{dt} = \vec{F}_{\text{tot}}.$$

We want to get rid of the vectors, but we also want the resulting equation to be useful. One very useful way to get rid of the vectors is to take the dot product of both sides of the equation with $d\vec{r}$ (this time, I'm using $\vec{r}$ instead of $\vec{x}$ to represent position) and to take the integral of both sides:

$$\int_{\vec{r}_i}^{\vec{r}_f} m\frac{d\vec{v}}{dt} \cdot d\vec{r} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{\text{tot}} \cdot d\vec{r}.$$

On the left-hand side, we can switch the $d\vec{v}$ with the $d\vec{r}$ to get

$$\int_{\vec{v}_i}^{\vec{v}_f} m\frac{d\vec{r}}{dt} \cdot d\vec{v} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{\text{tot}} \cdot d\vec{r}.$$

Remember that $\frac{d\vec{r}}{dt} = \vec{v}$, and so the integral on the left is

$$\int_{\vec{v}_i}^{\vec{v}_f} m\frac{d\vec{r}}{dt} \cdot d\vec{v} = \int_{\vec{v}_i}^{\vec{v}_f} m\vec{v} \cdot d\vec{v} = \left(\frac{1}{2}m\vec{v} \cdot \vec{v}\right)\Big|_{\vec{v}_i}^{\vec{v}_f} = \frac{1}{2}m|\vec{v}_f|^2 - \frac{1}{2}m|\vec{v}_i|^2.$$

Therefore

$$\boxed{\frac{1}{2}m|\vec{v}_f|^2 - \frac{1}{2}m|\vec{v}_i|^2 = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{\text{tot}} \cdot d\vec{r}.}$$

This is the work-energy theorem. On the left-hand side, there are no more vectors. There is the mass, the magnitude of the final velocity $|\vec{v}_f|$, and the magnitude of the initial velocity $|\vec{v}_i|$. We define the kinetic energy as

$$\boxed{K = \frac{1}{2}m|\vec{v}|^2.}$$

Then the work-energy theorem becomes

$$\Delta K = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{\text{tot}} \cdot d\vec{r}.$$

The right-hand side of this equation still looks very complicated. This integral is called the total work done by the force $\vec{F}_{\text{tot}}$. We often use the variable W to mean work:

$$\boxed{W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{\text{tot}} \cdot d\vec{r}.}$$

Using this definition, we can write the work-energy theorem simply as

$$\boxed{\Delta K = W.}$$

The integral to calculate work W is complicated, but it becomes much simpler in certain circumstances.

9.2.1 Conservative Forces

For certain types of forces, we will be able to calculate the work integral and get a simple result. These forces are called conservative forces. Gravity and the spring force given by Hooke's law both are both conservative forces. Friction is not a conservative force.

Hooke's Law
Because a spring which obeys Hooke's law gives a conservative force, we can calculate the work done by a spring. In one-dimension,

$$\boxed{W_{\text{spring}} = \int_{x_i}^{x_f} F_{\text{spring}} dx = \int_{x_i}^{x_f} (-kx) dx = -\frac{1}{2} k \Delta(x^2).}$$

Gravity
In one dimension, the work done by gravity is

$$W_{\text{grav}} = \int_{y_i}^{y_f} F_{\text{grav}} dy = \int_{y_i}^{y_f} (-mg) dy = -mg\Delta y.$$

Actually, this formula is also true in more than one dimension:

$$\boxed{W_{\text{grav}} = -mg\Delta y,}$$

where Δy is the change in the object's height.

10 Potential Energy and Energy Conservation

In the last lecture, we talked about conservative forces, with gravity and Hooke's law as examples. We found that

$$W_{\text{spring}} = -\frac{1}{2} k \Delta(x^2)$$

and

$$W_{\text{grav}} = -mg\Delta y.$$

For these conservative forces, the force-energy theorem becomes a lot simpler. For example, for gravity,

$$\Delta K = W_{\text{grav}} = -mg\Delta y.$$

If we write this out using the definition of ΔK and Δy, we get

$$\frac{1}{2} m |\vec{v}_f^2| - \frac{1}{2} m |\vec{v}_i^2| = -(mgy_f - mgy_i).$$

Notice that there are no vectors in this equation. We can rearrange this equation to show that

$$\frac{1}{2}m|\vec{v}_f^2| + mgy_f = \frac{1}{2}m|\vec{v}_i^2| + mgy_i.$$

Notice that both sides of this equation look similar. The only difference is that the left side is evaluated at the final time, and the right side is evaluated at the initial time. The work-energy theorem is telling us that the quantity $\frac{1}{2}m|\vec{v}^2| + mgy$ says the same no matter which time we evaluate it. This quantity is called the total energy.

More generally, for any conservative force, we can define a potential energy U in such a way that the total energy E, defined by

$$\boxed{E = K + U,}$$

stays the same at all points in time. In other words,

$$E_f = E_i.$$

As we just saw, the potential energy for gravity is

$$U_{\text{grav}} = mgy.$$

The potential energy for a spring that obey's Hooke's law is

$$U_{\text{spring}} = \frac{1}{2}kx^2.$$

11 Advanced Forces

11.1 Newton's Third Law

Remember that Newton's third law says that if object A causes a force on object B, then object B causes a force on object A that is equal in magnitude, but opposite in direction.

11.1.1 Stacked blocks

Activity: Suppose two blocks are stacked on top of each other. The coefficient of static friction between the two blocks is $\mu_s = 0.6$. There is a force $\vec{F}$ on the bottom block pushing it to the right. Draw free-body diagrams for each block.

Solution: For the top block, the only forces are gravity (pointing down), the normal force (pointing up) and the force of static friction which prevents the block from moving relative to the bottom block. Since the bottom block is being pushed to the right, the force of static friction on the top block must also point to the right so that the top block keeps up with the bottom block.

The bottom block experiences the force of gravity (pointing down) and the force $\vec{F}$ (pointing right). However, these are not the only forces. Because the top block experiences a normal force from the bottom block, by Newton's third law, the bottom block experiences a normal force from the top block of equal magnitude pointing down. Similarly, since the top block experiences a force from static friction pointing right, the bottom block experiences a force of equal magnitude pointing left.

11.2 Centripetal Forces

Earlier, we learned that an object in *uniform circular motion* (circular motion at a constant speed) has an acceleration that points toward the center of the circle and has magnitude

$$a_c = \frac{|\vec{v}|^2}{R},$$

where R is the radius of the circle. Based on Newton's second law, we know that this acceleration must be caused by some force. A force that causes an object to move in uniform circular motion is called a *centripetal force*. The centripetal force can come from many different sources. For example, the centripetal force could be caused by gravity, tension, spring forces, or normal forces. Whatever the source of the centripetal force is, we know that the direction of the centripetal force must be toward and the center of the circle, and its magnitude must be

$$F_c = \frac{m|\vec{v}|^2}{R}.$$

If either of these things were not true, then the object would not have the proper acceleration, and so it would not move in uniform circular motion.

11.2.1 Spinning with friction

Activity: A plate is spinning about its center at a constant speed. There is a block resting on the plate at a distance of 0.02 m from its center. The coefficient of static friction between the plate and the block is $\mu_s = 0.2$. Because the plate is spinning, the block is moving at a speed v. What is the maximum speed v before the block will fly off the plate?

Solution: Because the block is in uniform circular motion, its acceleration must have magnitude $\frac{|\vec{v}|^2}{R}$ and be pointed towards the center of the plate. In order to produce this acceleration, the force on the block must have magnitude

$$F_c = \frac{m|\vec{v}|^2}{R}$$

and be pointed toward the center of the plate. Start by drawing a free-body diagram for the block. The only forces acting on it are gravity, the normal force, and static friction. Gravity and the normal force are pointed along the up-down direction, and so they cannot produce acceleration toward the center of the plate. The only force that can produce this acceleration is static friction. The maximum value of static friction is

$$F_s \le \mu_s |\vec{N}|.$$

We need to find $\vec{N}$. Because the object is not accelerating in the up-down direction, the normal force and gravity must cancel each other out by Newton's second law. Therefore, $|\vec{N}| = mg$, and

$$F_s \le \mu_s mg.$$

To avoid flying off the plate, the force caused by friction must be $\frac{m|\vec{v}|^2}{R}$. Therefore, we must have

$$\frac{m|\vec{v}|^2}{R} \le F_{s,\max} = \mu_s mg$$
$$\implies |\vec{v}|^2 \le \mu_s gR$$
$$\implies |\vec{v}| \le \sqrt{\mu_s gR} = \sqrt{0.2 \cdot 9.8 \text{ m/s}^2 \cdot 0.02 \text{ m}} \approx 0.2 \text{ m/s}$$

11.2.2 Conical Pendulum

Activity: A conical pendulum is a pendulum that moves in a circle (when viewed from above). It is called conical because the string traces out a cone shape. If the string is 2 m long and the weight at the end of the string moves in a circle of radius 0.7 m at a constant speed, what is the period of the pendulum (the amount of time it takes to complete one circle)?

Solution: There are several steps to this problem. One way to find the period would be to first find the speed of the pendulum. To find the speed, we can use the fact that the weight is moving in uniform circular motion. That means

$$|\vec{a}| = \frac{|\vec{v}|^2}{R}$$
$$\implies |\vec{v}| = \sqrt{|\vec{a}|R}$$

We know $R = 0.7$ m from the description of the problem. We don't yet know $|\vec{a}|$, but we can try to figure this out using Newton's second law. First, draw a free-body diagram for the weight. The only two forces are gravity and the tension from the string. If we define the y axis to point up, and the x axis to point toward the center of the circle, then for gravity

$$F_{g,x} = 0 \text{ and } F_{g,y} = -mg.$$

For the tension force, we need some trigonometry. If the length of the string is 2 m, and the distance of the weight from the center of the circle is 0.7 m, then we can form a right triangle with 2 m as the hypotenuse and a vertical line as one side. The angle θ of the string above the horizontal direction is

$$\theta = \arccos\left(\frac{0.7 \text{ m}}{2 \text{ m}}\right) \approx 69.5^\circ.$$

Therefore, the tension force $\vec{T}$ has x and y components

$$T_x = |\vec{T}| \cos 69.5^\circ \text{ and } T_y = |\vec{T}| \sin 69.5^\circ.$$

From Newton's second law for the y components,

$$T_y + F_{g,y} = ma_y = 0$$

$$\implies |\vec{T}| \sin 69.5^\circ - mg = 0$$

$$\implies |\vec{T}| = \frac{mg}{\sin 69.5^\circ}.$$

From Newton's second law for the x components,

$$T_x + T_{g,x} = ma_x$$

$$\implies |\vec{T}| \cos 69.5^\circ + 0 = ma_x$$

$$\implies a_x = \frac{|\vec{T}| \cos 69.5^\circ}{m} = \frac{mg \cos 69.5^\circ}{m \sin 69.5^\circ} = \frac{g \cos 69.5^\circ}{\sin 69.5^\circ} \approx 3.66 \text{ m/s}^2.$$

If we put this into our formula for $|\vec{v}|$, we get

$$|\vec{v}| = \sqrt{|\vec{a}|R} \approx \sqrt{3.66 \text{ m/s}^2 \cdot 0.7 \text{ m}} \approx 1.6 \text{ m/s}.$$

Now we can find the period. Since the circle has a circumference of $2\pi R$, the amount of time it takes to complete one circle is

$$T = \frac{\text{distance}}{\text{speed}} = \frac{2\pi R}{1.6 \text{ m/s}} \approx 2.7 \text{ s}.$$

12 Power

Power is defined as the rate at which work is done. Mathematically,

$$\boxed{P = \frac{d}{dt}W \equiv \frac{d}{dt}\int_{\text{path}} \vec{F} \cdot d\vec{r}.}$$

Large power means that work is being done quickly, while low power means work is being done slowly. The average power is given by the formula

$$\boxed{P_{\text{av}} = \frac{W}{t},}$$

where W is the total work done, and t is the amount of time it took to do this work. If the work is constant, then this formula gives the exact power.

The units of power are Joules per second (J/s). These are the units of work/energy (Joules) divided by units of time (seconds). Joules per second are also called Watts (abbreviated "W").

$$\mathrm{W} \equiv \frac{\mathrm{J}}{\mathrm{s}} \equiv \frac{\mathrm{kg} \cdot \mathrm{m}^2}{\mathrm{s}^3}.$$

12.1 Force Times Velocity

When $\vec{F} \cdot d\vec{r}$ is constant along the object's path, we found that the amount of work done is

$$W = |\vec{F}_{\text{const}}| \Delta r \cos\theta,$$

where θ is the angle between the force and the direction in which the object is moving. In this case, the power is

$$\boxed{P = \frac{d}{dt} W = |\vec{F}_{\text{const}}| \frac{d}{dt} \Delta r \cos\theta = |\vec{F}_{\text{const}}| |\vec{v}| \cos\theta.}$$

12.2 Car Problem

Activity: A car of mass 4000 kg starts at rest and then accelerates at a constant rate of 10 m/s^2. What is the power of the car as a function of time?

Solution: The force of the car is given by Newton's second law:

$$F = ma = 4000 \text{ kg} \cdot 10 \text{ m/s}^2 = 40000 \text{ N}.$$

The velocity is

$$v = v_0 + at = 0 + 10 \text{ m/s}^2 \cdot t.$$

Since the force is constant and the car doesn't change directions, $\vec{F} \cdot d\vec{r}$ is constant. Therefore,

$$P = |\vec{F}_{\text{const}}| |\vec{v}| \cos\theta = (40000 \text{ N})(10 \text{ m/s}^2 \cdot t) \cos 0^\circ = 400000 \text{ N} \cdot \text{ m/s}^2 \cdot t$$

$$= 400000 \frac{\text{kg} \cdot \text{ m}}{\text{s}^2} \cdot \text{ m/s}^2 \cdot t = 400000 \frac{\text{kg} \cdot \text{ m}^2}{\text{s}^4} \cdot t$$

13 Center of Mass

Intuitively, the center of mass is supposed to be the center of an object (or a system of multiple objects), but it is weighed to give more importance to objects that are more massive.

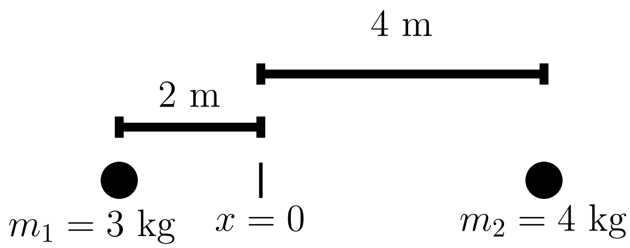


Figure 43: Example of a 1D center-of-mass problem. The center of mass is located at $x_{\text{COM}} = \frac{(3\text{ kg})(-2\text{ m})+(4\text{ kg})(4\text{ m})}{3\text{ kg}+4\text{ kg}} \approx 1.43$ m

13.1 One Dimension

13.2 Two Points

Suppose we have two massive points. One of them has mass m_1 and is located at x_1, and the other has mass m_2 and is located at x_2. Then the center of mass, x_{COM}, is defined as

$$x_{\text{COM}} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

x_{COM} will lie somewhere in between x_1 and x_2. More specifically:

- If the masses are the same ($m_1 = m_2$), then x_{COM} will be exactly in the middle between x_1 and x_2.
- If m_1 is bigger than m_2, then x_{COM} will still be in between, but it will be closer to x_1.
- If m_2 is bigger than m_1, then x_{COM} will still be in between, but it will be closer to x_2.

See Figure 43 for an example.

13.2.1 Multiple Points

Suppose we have multiple massive points with masses m_1, m_2, m_3 etc. and locations x_1, x_2, x_3, etc. Then the center of mass is defined as

$$\boxed{x_{\text{COM}} \equiv \frac{\sum_i m_i x_i}{\sum_i m_i}} \equiv \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}.$$

13.3 Multiple Dimensions

In two or three dimensions, the center-of-mass equation above holds **for each coordinate separately**:

$$x_{\text{COM}} \equiv \frac{\sum_i m_i x_i}{\sum_i m_i}.$$

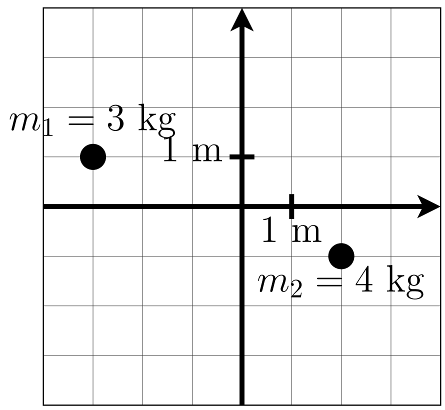


Figure 44: Example of a 2D center-of-mass problem. The center of mass is located at x-coordinate $x_{\text{COM}} = \frac{(3\text{ kg})(-3\text{ m})+(4\text{ kg})(2\text{ m})}{3\text{ kg}+4\text{ kg}} \approx -0.143$ m and y-coordinate $y_{\text{COM}} = \frac{(3\text{ kg})(1\text{ m})+(4\text{ kg})(-1\text{ m})}{3\text{ kg}+4\text{ kg}} \approx -0.143$ m (which happens to be the same as the x-coordinate).

$$y_{\text{COM}} \equiv \frac{\sum_i m_i y_i}{\sum_i m_i}.$$

$$z_{\text{COM}} \equiv \frac{\sum_i m_i z_i}{\sum_i m_i}.$$

See Figure 44 for an example.

13.4 Symmetric Objects

For a massive object like a ball or a cube that is not just a single point, we would in general have to use calculus to find the center of mass. However,

- if the object has some symmetry so that it's clear where its center is,
- and if the mass is uniformly distributed throughout the object,

then the center of mass is just the center of the object. See Figure 45 for some examples.

13.5 Superposition Principle

Suppose we have multiple objects (for example, a square and a cube). To find the center of mass of the whole system we just need to find the center of mass of each object individually $\vec{x}_{\text{COM},1}$, $\vec{x}_{\text{COM},2}$, etc. and then combine them using the center-of-mass formula:

$$\vec{x}_{\text{COM}} \equiv \frac{\sum_i M_i \vec{x}_{\text{COM},i}}{\sum_i M_i},$$

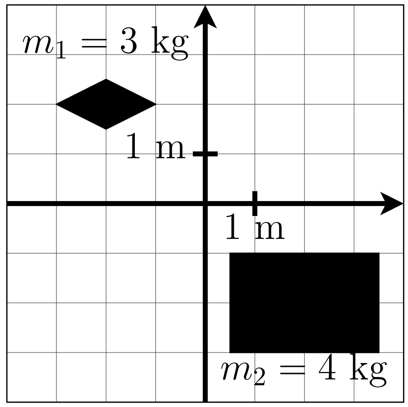


Figure 45: Example of center-of-mass problems with extended objects. Assuming the mass is distributed uniformly, the center of mass of the diamond is in the center of the diamond at $x_{\text{COM}} = -2$ m and $y_{\text{COM}} = 2$ m. The center of mass of the rectangle is located at its center at $x_{\text{COM}} = 2$ m and $y_{\text{COM}} = -2$ m. The center of mass of the entire system (the diamond and the rectangle together) is given by the superposition principle: $x_{\text{COM}} = \frac{(3\text{ kg})(-2\text{ m})+(4\text{ kg})(2\text{ m})}{3\text{ kg}+4\text{ kg}} \approx 0.286$ m and $y_{\text{COM}} = \frac{(3\text{ kg})(2\text{ m})+(4\text{ kg})(-2\text{ m})}{3\text{ kg}+4\text{ kg}} \approx -0.286$ m

where M_i is the total mass of object i and $x_{\text{COM},i}$ is the center of mass of object i **by itself**. See Figure 45 for an example.

13.6 Dynamics of Center of Mass

We can rewrite the center-of-mass equation as

$$M\vec{x}_{\text{COM}} = \sum_i m_i \vec{x}_i$$

If we take the time derivative of both sides of this equation, we get (assuming mass is constant)

$$\frac{d}{dt}\left(M\vec{x}_{\text{COM}}\right) = \frac{d}{dt}\left(\sum_i m_i \vec{x}_i\right) \implies M\frac{d\vec{x}_{\text{COM}}}{dt} = \sum_i m_i \frac{d\vec{x}_i}{dt}$$
$$\implies M\vec{v}_{\text{COM}} = \sum_i m_i \vec{v}_i.$$

If we take the time derivative again, we get

$$M\vec{a}_{\text{COM}} = \sum_i m_i \vec{a}_i.$$

In these equations, v_{COM} tells us how fast the center of mass of the system is moving (each part of the system may be moving with different velocities $\vec{v}_i$, but the center of mass is moving at velocity $\vec{v}_{\text{COM}}$). Similarly, $\vec{a}_{\text{COM}}$ tells us how fast the center of mass is accelerating.

Now we can use Newton's Second Law:

$$M\vec{a}_{\text{COM}} = \sum_i m_i \vec{a}_i = \sum_i \vec{F}_i$$

This second sum is now a sum over all forces acting on any object in the system (remember, there can be more than one force acting on each object). This gives us a way to find out how fast a system is accelerating. Actually, we can make this even simpler! Remember that Newton's Third Law says that whenever one object exerts a force on another object, there is an equal but opposite force. That means that a lot of the forces in the sum $\sum_i \vec{F}_i$ actually cancel each other out. At the end of the day, the only force that do not get canceled out are **external** forces. These are forces that are caused by objects *outside* of the system.

Example: (See Figure 46) Imagine a system of two balls connected to each other by a spring and falling towards the ground. When ball 1 moves, it might stretch the spring and cause a force on ball 2. Ball 2 is also stretching the spring, and so it exerts an equal but opposite force on ball 1. The spring forces are *internal* forces: they are caused by objects within the system. Gravity is an *external* force: it is cause by the earth, which is not part of the system as we have defined it. Therefore, the acceleration of the center of mass will be

$$M\vec{a}_{\text{COM}} = \sum_i \vec{F}_{i,\text{external}} = \vec{F}_{g,\text{ball 1}} + \vec{F}_{g,\text{ball 2}}$$

The only external forces are the force of gravity on ball 1 and the force of gravity on ball 2.

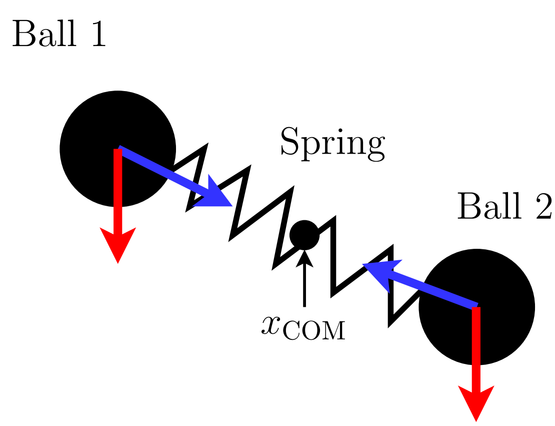


Figure 46: The spring forces (blue) cancel each other out, leaving only the external gravitational forces (red) on ball 1 and ball 2.

To summarize, we have found that the acceleration of the center of mass of a system is time the total mass M is given by the sum of the external forces acting on objects in the system:

$$\boxed{M\vec{a}_{\mathrm{COM}} = \sum_i \vec{F}_{i,\mathrm{ext}}.}$$

14 Momentum and Inelastic Collisions

14.1 System Equilibrium

What if there are no external forces acting on a system? Then

$$M\vec{a}_{\mathrm{COM}} = \sum_i \vec{F}_{i,\mathrm{ext}} = 0.$$

This means that $\vec{a}_{\mathrm{COM}} = 0$ and $\vec{v}_{\mathrm{COM}}$ must be constant. When there are no external forces, the center of mass moves at a constant velocity.

Example: (See Figure 47) Imagine a person standing at one end of a canoe that is at rest. The person then walks toward the other end of the canoe. Assuming there are no external forces acting on the canoe (the water is perfectly still and frictionless) what will happen to the canoe? Initially, the canoe and person are at rest. If we treat them as a system, their center of mass must initially have velocity $\vec{v}_{\mathrm{COM}} = 0$. Since there are no external forces, $\vec{v}_{\mathrm{COM}}$ must remain constant at 0. In other words, the center of mass does not move. Therefore, when the person moves across to the other side of the canoe, the canoe must move in the opposite direction so that all together, their center of mass stays in the same place.

14.2 Momentum

We will now briefly change the topic to discuss momentum. The momentum of an object is defined as

$$\vec{p} = m\vec{v}.$$

Note that momentum is a vector. If we take the time derivative of this equation, we get (assuming mass is constant),

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a} = \sum_i \vec{F}_i.$$

Actually, $\frac{d\vec{p}}{dt} = \sum_i \vec{F}_i$ holds even if mass is not constant and is a more correct form of Newton's Second Law.

For a system of multiple objects, we can define the total momentum $\vec{p}$ as the sum of the momenta of each individual object in the system:

$$\vec{p} = \sum_i \vec{p}_i. = \sum_i m_i\vec{v}_i = M\vec{v}_{\mathrm{COM}}.$$

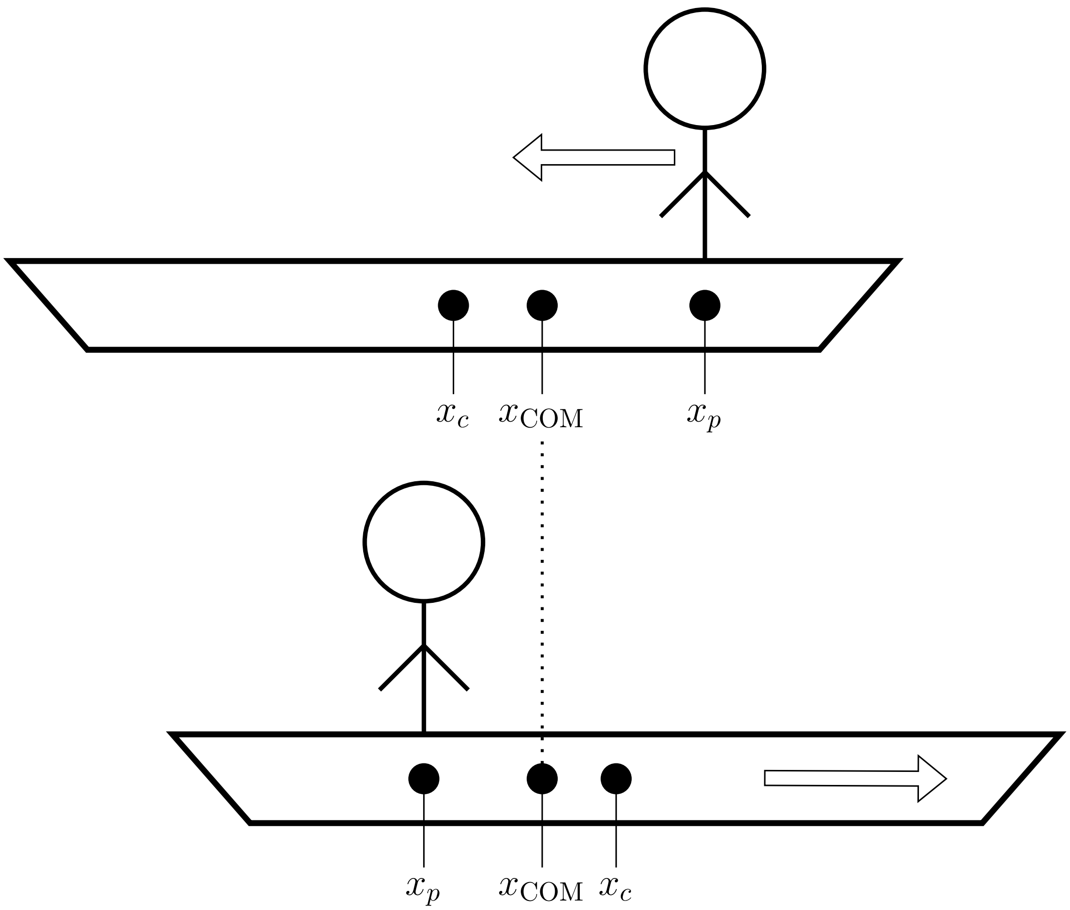


Figure 47: The x-components of the center of mass of the person x_p and of the canoe x_c are marked. As the person moves toward the other side of the canoe, the canoe must move in the opposite direction so that the total center of mass x_{COM} remains unmoving.

If there are no external forces on the system, then the total momentum $\vec{p}$ is constant (because $\vec{v}_{\text{COM}}$ is constant). This will turn out to be a very useful fact for solving collision problems.

Practice problem: An astronaut of mass 80 kg is stranded in space outside of the spaceship. The astronaut is initially at rest relative to the spaceship. The astronaut throws a wrench of mass 2 kg with velocity 10 m/s in the direction away from the space ship.

- What is the initial total momentum of the astronaut-and-wrench system?
- What is the final momentum of the wrench?
- What is the final momentum of the astronaut?
- How fast and in what direction is the astronaut moving after throwing the wrench?

Answer: The astronaut and wrench initially have no velocity, and so their total momentum is zero. We will choose the direction away from the spaceship to be negative and towards the spaceship to be positive. Therefore, after being thrown, the wrench has a momentum $p_w = m_w v_w = -(2\text{ kg})(10\text{ m/s}) = -20\text{kg}\cdot\text{m/s}$. Now we need to find the momentum of the astronaut p_a. The total momentum $p_{\text{total}} = p_w + p_a$ must be constant, since there are no external forces. Since the initial total momentum was zero, we conclude $0 = p_{\text{total}} = p_w + p_a$. Therefore, $p_a = -p_w = +20\text{kg}\cdot\text{m/s}$. By definition, $p_a = m_a v_a$. We can solve this equation for the astronaut's final velocity:

$$v_a = \frac{p_a}{m_a} = \frac{20\text{kg}\cdot\text{m/s}}{80\text{kg}} = 0.25\text{ m/s}.$$

The speed is 0.25 m/s and the direction is towards the spaceship (since v_a is positive).

14.3 Collisions and Explosions

Imagine a process in which multiple objects collide. If there are no external forces, the the total momentum of the colliding objects is conserved. For example, suppose two cars crash into each other. If neither driver applies the brakes, we can probably ignore external forces like friction. Then the total momentum of the cars will be conserved in the collision.

An explosion is just a time-reversed collisions (see Figure 48). If there are no external force, momentum will also be conserved in an explosion. For example, when a firework explodes, the exploded pieces will continue moving upwards. If we ignore external forces like gravity, the center of mass of all of these piece will continue moving at the same velocity the firework had before the explosion.

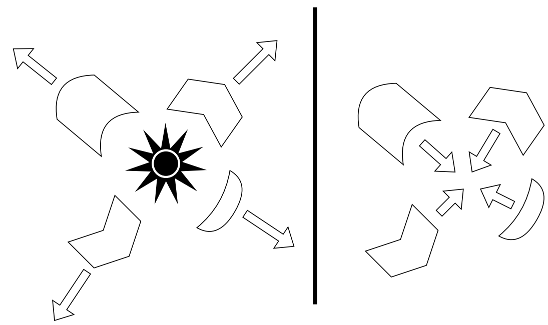


Figure 48: In an explosion, objects come apart. In a collision, objects come together.

14.3.1 Inelastic Collisions

Sometimes knowing that the total momentum is conserved is enough information to completely solve a collision or explosion problem, but sometimes we need more information. In an **elastic collision**, both momentum and energy are conserved. We will talk more about elastic collisions later.

A collision in which energy is not conserved is called an **inelastic collision**. In this course, the main examples of inelastic collisions are collisions where the objects stick together after colliding. Collisions in which the objects stick together are always inelastic collisions. For example, if a ball collides with a lump of clay and sticks to it, this is an inelastic collision. The opposite of this kind of inelastic collision is an explosion: the objects initially stick together before exploding apart. In these kinds of problems, we can use conservation of momentum to predict the outcome of the collision/explosion.

Practice Problem: A bullet of mass 0.01 kg is shot into a block of wood of mass 2 kg and becomes stuck in the block. Initially, the block is at rest. After the bullet becomes stuck in it, it starts moving at a speed of 2 m/s. What is the initial velocity of the bullet?

Answer: This is a typical inelastic collision problem. The initial momentum of the system comes only from the bullet (since the block is initially at rest)

$$p_{\text{initial}} = m_{\text{bullet}} v_{i,\text{bullet}}.$$

For the final momentum, both the bullet and block are moving. Since the bullet is stuck in the block, they are both moving *at the same velocity.* This is an important point in inelastic collision problems when objects stick together. Let's call the final velocity of the bullet and the block v_f. Then

$$p_f = m_{\text{bullet}} v_f + m_{\text{block}} v_f.$$

This must be equal to the initial momentum:

$$p_i = p_f$$

$$\implies m_{\text{bullet}} v_{i,\text{bullet}} = m_{\text{bullet}} v_f + m_{\text{block}} v_f$$

Solving for $v_{\text{i,bullet}}$, we get

$$v_{i,\text{bullet}} = \frac{m_{\text{bullet}} v_f + m_{\text{block}} v_f}{m_{\text{bullet}}} = \frac{(0.01 \text{ kg})(2 \text{ m/s}) + (2 \text{ kg})(2 \text{ m/s})}{0.01 \text{ kg}}$$

$$= 402 \text{ m/s}.$$

Practice Problem: As shown in Figure 49, a plate of mass $10kg$ which is initially at rest explodes into three pieces. One of the pieces with mass $m_1 = 2$ kg travels with a velocity of magnitude $|\vec{v}_1| = 94$ m/s at an angle of $40°$ to the left of the negative vertical. Another piece of mass $m_2 = 5$ kg travels with a velocity of magnitude $|\vec{v}_2| = 65$ m/s straight up. What is the velocity of the last piece $\vec{v}_3$?

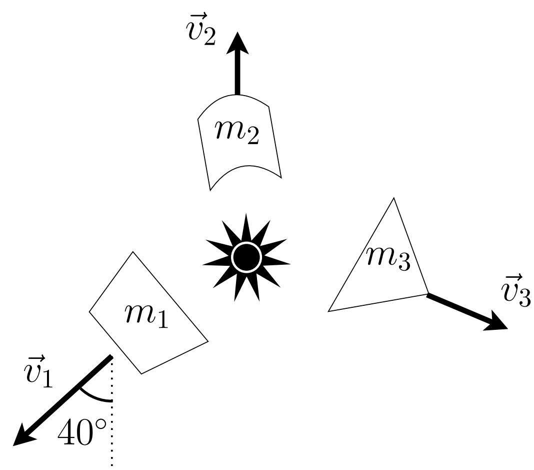


Figure 49: A plate explodes into three pieces.

Answer: This is a typical explosion problem. Since the plate was initially at rest, the initial total momentum if $\vec{p}_i = 0$. After the explosion, we can find the momentum of m_1 and m_2. Remember that momentum is a vector, and so in this two-dimensional problem, we need to calculate x and y components separately. As usual, we will choose up as the positive y direction and right as the positive x direction. To break up $\vec{v}_1$ into components, we note that $\vec{v}_1$ makes an angle of $50°$ below the negative x-axis, and that both the x and y components should be negative.

$$p_{1,x} = m_1 v_{1,x} = -m_1|\vec{v}_1|\cos(50°) = -(2 \text{ kg})(94 \text{ m/s})\cos(50°) \approx -121 \text{ kg} \cdot \text{m/s}$$

$$p_{1,y} = -m_1|\vec{v}_1|\sin(50°) = \approx -144 \text{ kg} \cdot \text{m/s}$$

For m_2, the velocity is directly upward, and so

$$p_{2,x} = m_2 \vec{v}_{2,x} = 0$$

$$p_{2,y} = m_2\vec{v}_{2,y} = (5 \text{ kg})(65 \text{ m/s}) = 325 \text{ kg}\cdot\text{m/s}$$

Now we can find $\vec{p}_3$ using the fact that momentum is constant (assuming no external forces are present):

$$0 = \vec{p}_i = \vec{p}_f = \vec{p}_1 + \vec{p}_2 + \vec{p}_3$$

Like all vector equations, this equation must hold for both the x and y components:

$$p_{1,x} + p_{2,x} + p_{3,x} = 0$$

$$\implies p_{3,x} = -p_{1,x} - p_{2,x} = -(-121 \text{ kg}\cdot\text{m/s}) - 0 = 121 \text{ kg}\cdot\text{m/s}$$

and similarly

$$p_{1,y} + p_{2,y} + p_{3,y} = 0$$

$$\implies p_{3,y} = -p_{1,y} - p_{2,y} = -(-144 \text{ kg}\cdot\text{m/s}) - (325 \text{ kg}\cdot\text{m/s}) = -181 \text{ kg}\cdot\text{m/s}$$

Now we can find $\vec{v}_3$ using

$$\vec{p}_3 = m_3\vec{v}_3.$$

Since the initial plate had mass 10 kg, we conclude that

$$m_3 = 10 \text{ kg} - m_1 - m_2 = 10 \text{ kg} - 2 \text{ kg} - 5 \text{ kg} = 3 \text{ kg}.$$

Then

$$v_{3,x} = \frac{p_{3,x}}{m_3} = \frac{121 \text{ kg}\cdot\text{m/s}}{3 \text{ kg}} \approx 40.3 \text{ m/s}$$

and

$$v_{3,y} = \frac{p_{3,y}}{m_3} = \frac{-181 \text{ kg}\cdot\text{m/s}}{3 \text{ kg}} \approx -60.3 \text{ m/s}.$$

15 Advanced Momentum

15.1 Inelastic Collisions

We can prove that collisions in which objects stick together are inelastic. Imagine two blocks with masses m_1 and m_2 and velocities v_1 and v_2 that collide with each other and stick together. The initial momentum is

$$p_i = m_1v_1 + m_2v_2.$$

and the final momentum is

$$p_f = m_1v_f + m_2v_f$$

(remember, v_f is the same for both blocks because they stick together.) Since momentum in conserved in collisions (as long as there are no external forces),

$$p_i = p_f$$

$$\implies m_1v_1 + m_2v_2 = m_1v_f + m_2f_f$$
$$\implies v_f = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}.$$

Now the initial kinetic energy is

$$K_i = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2.$$

On the other hand, the final kinetic energy is

$$K_f = \frac{1}{2}(m_1 + m_2)v_f^2 = \frac{1}{2}(m_1 + m_2)\frac{(m_1v_1 + m_2v_2)^2}{(m_1 + m_2)^2}$$

If we then subtract K_i in order to get ΔK, we get (after a bit of algebra)

$$\Delta K = -\frac{m_1m_2}{2(m_1 + m_2)}(v_1 - v_2)^2.$$

This can only be zero if $v_1 = v_2$. However, if $v_1 = v_2$, then both objects must have initially been moving in the same direction at the same speed. This means they never could have collided with each other in the first place! Therefore, any time two objects stick together after colliding, the total energy is not conserved.

Practice Problem: In a ballistic pendulum, a bullet of mass m is shot into a block of mass M and becomes stuck in it. The block is hung so that it can swing backwards after the bullet hits it. After getting hit with the bullet, the block rises to a maximum height H. What is the velocity of the bullet in terms of m, M, H, and g?

Solution: Let's call the initial velocity of the bullet v_b. If the block starts at rest, the initial momentum of the system comes only from the bullet:

$$p_i = mv_b.$$

After the bullet hits the block, the bullet and the block become stuck together and move with velocity v. Since this is an inelastic collision problem, the momentum must still be the same:

$$p_i = p_f = (M + m)v.$$

We can solve this for the initial velocity of the bullet:

$$mv_b = (M + m)v \implies v_b = \frac{(M + m)v}{m}.$$

This formula still has v in it though. We need a way to find v in terms of H and g.

A key phrase from the problem statement was "maximum height." When the pendulum reaches its maximum height, it must have velocity 0. Therefore, at this point, all its energy is potential. The only two forces acting are gravity and the tension of the strings. The tension force is always perpendicular to the motion of the object and does not do any work. That means we only need gravitational kinetic energy. The final energy of the block (with the bullet inside it) is

$$E_f = (M + m)gH$$

Right after the bullet hits the block, the total energy comes only from kinetic energy (since it starts at height 0). Therefore,

$$E_i = \frac{1}{2}(M + m)v^2.$$

Setting $E_i = E_f$, we can solve for the velocity of the block right after the bullet hits it:

$$\frac{1}{2}(M + m)v^2 = (M + m)gH \implies v^2 = 2gH \implies v = \sqrt{2gH}.$$

Therefore,

$$v_b = \frac{(M + m)v}{m_b} = \frac{(M + m)\sqrt{2gH}}{m}$$

15.2 Elastic Collisions

In an elastic collision, both energy and momentum are conserved. We can use the equations

$$p_i = p_f$$

and

$$E_i = E_f$$

to solve elastic collision problems.

Practice Problem: Two balls collide elastically. If ball 1 began at rest and has mass m, and ball 2 was moving with velocity $v = 5$ m/s and has mass $2m$, what is the final velocity of each ball?

Solution: The initial momentum is

$$p_i = 0 + 2mv = 2mv.$$

The final momentum is

$$p_f = mv_1 + 2mv_2,$$

where v_1 and v_2 are the final velocities of ball 1 and ball 2 (which are unknown). The momentum conservation equation $p_i = p_f$ then gives us

$$2mv = mv_1 + 2mv_2.$$

$$\implies 2v = v_1 + 2v_2.$$

Since there are two unknowns v_1 and v_2, we need another independent equation. The initial energy is

$$E_i = \frac{1}{2} \cdot 2mv^2 + 0 = mv^2$$

and the final energy is

$$E_f = \frac{1}{2}(mv_1^2 + 2mv_2^2).$$

The conservation of energy equation $E_i = E_f$ gives us

$$mv^2 = \frac{1}{2}(mv_1^2 + 2mv_2^2)$$

$$\implies 2v^2 = v_1^2 + 2v_2^2.$$

We now just need to do some algebra. We can solve the momentum equation for v_1 in terms of v and v_2:

$$v_1 = 2v - 2v_2.$$

Now we can replace v_1 in the energy equation with $2v - 2v_2$:

$$2v^2 = (2v - 2v_2)^2 + 2v_2^2 = 4v^2 - 8v_2v + 4v_2^2 + 2v_2^2$$

$$\implies 6v_2^2 - 8v_2v + 2v^2 = 0$$

$$\implies 3v_2^2 - 4v_2v + v^2 = 0$$

Putting in $v = 5$ m/s,

$$3v_2^2 - (20 \text{ m/s})v_2 + 25 \text{ m}^2/\text{s}^2 = 0.$$

This is a quadratic equation for v_2. By the quadratic formula:

$$v_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{(20 \text{ m/s}) \pm \sqrt{400 \text{ m}^2/\text{s}^2 - 4 \cdot 3 \cdot (25 \text{ m}^2/\text{s}^2)}}{2 \cdot 3}$$

$$= \frac{20 \text{ m/s} \pm 10\text{m/s}}{6} = 5 \text{ m/s or } 1.67 \text{ m/s}.$$

One of the solutions is $v_2 = 5$ m/s. This is just the velocity ball 2 had *before* the collision. That means we must choose the other solution, $v_2 = 1.67$ m/s. Then we can put this back into the momentum equation to find v_1:

$$v_1 = 2v - 2v_2 = 6.66 \text{ m/s}.$$

15.3 Impulse

When no forces act on an object, its momentum remains constant. However, when there are forces present, the momentum changes. By Newton's Second Law, as we discussed in lecture 7.2, the rate of change of momentum is equal to the sum of all the forces acting on the object:

$$\frac{d\vec{p}}{dt} = \sum_i \vec{F}_i.$$

If we integrate this equation, we get

$$\Delta\vec{p} = \int \left(\sum_i \vec{F}_i\right) dt.$$

The integral of the total force on an object is called the **impulse**. We usually use the variable J for impulse (because the letter I will be used for something else later in the course).

$$\vec{J} \equiv \int \left(\sum_i \vec{F}_i\right) dt.$$

Practice Problem: A bug gets hit by a car. If the impulse of the car on the bug is 0.001 kg·m/s, what is the impulse of the bug on the car?

Solution: By Newton's Third Law, the force of the bug on the car is the same magnitude as the force of the car on the bug (but in the opposite direction). Therefore, when we integrate the force to get the impulse, we will get the same magnitude but opposite direction: -0.001 kg·m/s.

15.3.1 Constant Force

If the force is constant, then its impulse is

$$J = \int \vec{F}_{\text{const}} dt = \vec{F}_{\text{const}} \Delta t$$

Even if the force is not constant, we can get the *average* force on an object over a period of time Δt from the impulse:

$$\vec{F}_{\text{average}} = \frac{\vec{J}}{\Delta t}.$$

Practice Problem: A bike of mass 30 kg moving at 10 m/s crashes into a brick wall and comes to a complete stop.

- What is the total impulse on the car?
- If the crash takes 0.01 s, what is the average force on the bike?
- If the bike had instead crashed into a pillow and came to a stop in 0.2 s, what would the average force on the bike be?

Solution: If we define the direction of the bike's initial motion to be the positive direction, then the change in momentum is

$$\Delta p = p_f - p_i = 0 - 30 \text{ kg} \cdot 10 \text{ m/s} = -300 \text{ kg} \cdot \text{m/s}.$$

Therefore, the impulse must be

$$J = -300 \text{ kg} \cdot \text{m/s}.$$

The average force is

$$\vec{F}_{\text{average}} = \frac{\vec{J}}{\Delta t} = \frac{-300 \text{ kg} \cdot \text{m/s}}{0.01 \text{ s}} = -30,000 \text{ kg} \cdot \text{m/s}^2 = -30,000 \text{ N}.$$

If the bike instead crashed into a pillow, the average force would be

$$\vec{F}_{\text{average}} = \frac{\vec{J}}{\Delta t} = \frac{-300 \text{ kg} \cdot \text{m/s}}{0.2 \text{ s}} = -1,500 \text{ N}.$$

16 Rotational Kinematics

16.1 Arc Length

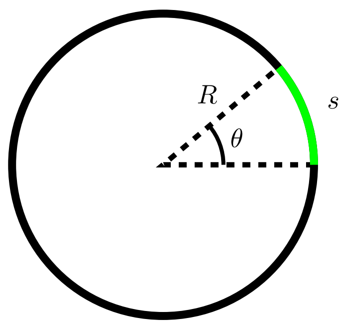


Figure 50: The *arc length* of the green part of the circle is $s = \theta R$, where θ is the angle measured in *radians* (not degrees).

Arc length can be calculated as shown in Figure 50 using the formula

$$\boxed{s = \theta R.}$$

This formula only holds if theta is measured in *radians*. As a quick way to check this formula, note that if $\theta = 2\pi$ radians (360°), then the we get

$$s = 2\pi R,$$

which is just the circumference of the entire circle as it should be.

Practice Problem: A wheel of radius 2 m is rolling without slipping along the ground. After it completes one rotation, how far has it traveled?

Solution: After completing one rotation, the wheel has rolled through its entire circumference of $2\pi R \approx 12.57$ m. If the wheel rolled without slipping, then the distance traveled along the ground must be the same.

16.2 Angular Velocity

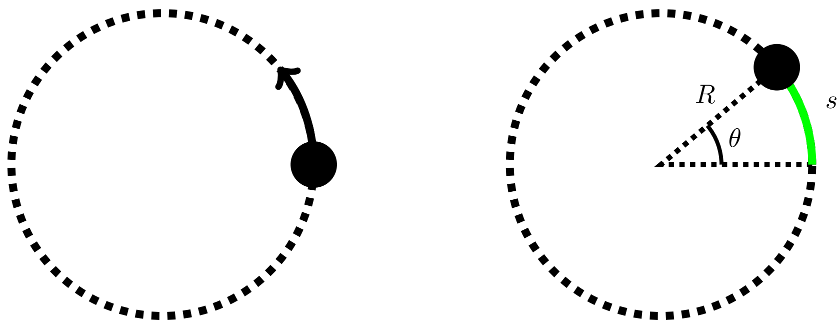


Figure 51: A point mass is traveling in a circle of radius R. After a certain amount of time, it covers a distance s. This distance is related to the angle θ by $s = \theta R$.

Imagine that some object is traveling in a circle of radius R as shown in Figure 51. After a certain amount of time, it will cover some distance s. This distance is related to the angle θ by $s = \theta R$. We can rearrange this formula to get the "angular distance" that object has traveled:

$$\theta = \frac{s}{R}.$$

We can then define an "angular velocity" by taking the time derivative of both sides of the equation:

$$\frac{d\theta}{dt} = \frac{1}{R}\frac{ds}{dt}.$$

Now $\frac{ds}{dt}$ is the derivative of the distance the object has traveled, which is just the speed of the object $|\vec{v}|$.

$$\frac{d\theta}{dt} = \frac{|\vec{v}|}{R}.$$

We will use the variable ω to represent **angular velocity**:

$$\boxed{\omega \equiv \frac{d\theta}{dt} = \frac{|\vec{v}|}{R}}.$$

When an extended object rotates, all points in the object rotate at the same *angular* velocity, although them may have different velocities (see Figure 52). Normal velocity is sometimes called **linear velocity** to distinguish it from angular velocity.

16.2.1 Direction of Angular Velocity

The formula above ($\omega = |\vec{v}|/R$) gives the *magnitude* of angular velocity. Angular velocity is also defined to have a direction. The direction of $\vec{\omega}$ tells us

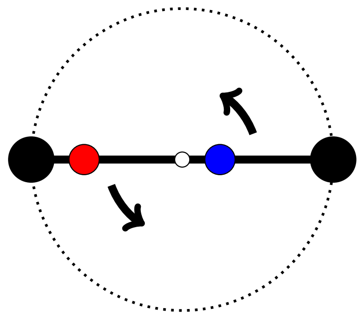


Figure 52: **Practice Problem:** A bar with black, red, and blue balls attached to it rotates as shown in the figure. Because all the objects are attached together, they must move at the same angular velocity. Rank the speed of the black, red, and blue points from lowest to highest.

Solution: From the angular velocity formula, $|\vec{v}| = \omega R$. ω is the same for each point, but they are each rotating at a different radius R. The black points are furthest from the center of the circle, so they have the largest R and therefore the fastest speed. The blue point has the lowest R and the slowest speed.

which direction the object is rotating (in 2D, the only options are clockwise and counterclockwise, but in 3D, an object can rotate in various directions). To find the direction of angular velocity, we use the **right hand rule**. If you curl your fingers in the direction of rotation, your thumb will naturally point in the direction of angular velocity (see Figure 53). Make sure to use your *right* hand and not your left.

Figure 53: The right hand rule: curl the fingers of your *right* hand in the direction of rotation as shown. Then your thumb points in the direction of $\vec{\omega}$.

16.3 Angular Acceleration

We can take the derivative of angular velocity to get **angular acceleration**. We will use the variable $\vec{\alpha}$ for angular acceleration:

$$\boxed{\vec{\alpha} = \frac{d\vec{\omega}}{dt}}$$

If we put in the definition of ω into the equation above, we get

$$|\vec{\alpha}| = \frac{d}{dt}\frac{|\vec{v}|}{R}.$$

If we assume the object is moving in a circle, then R must be constant, and so

$$|\vec{\alpha}| = \frac{1}{R}\frac{d|\vec{v}|}{dt}.$$

Note that $\vec{\alpha}$ is a vector. If the object is moving in a circle, the direction of $\vec{\alpha}$ follows a simple rule. When ω is increasing in magnitude, $\vec{\alpha}$ will have the same direction as $\vec{\omega}$. When ω is decreasing in magnitude, $\vec{\alpha}$ will have the opposite direction.

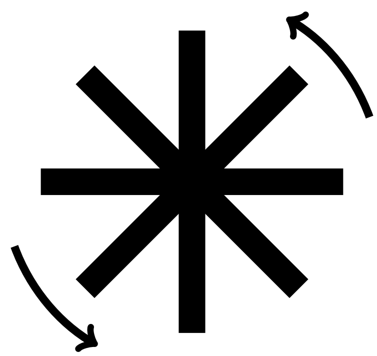
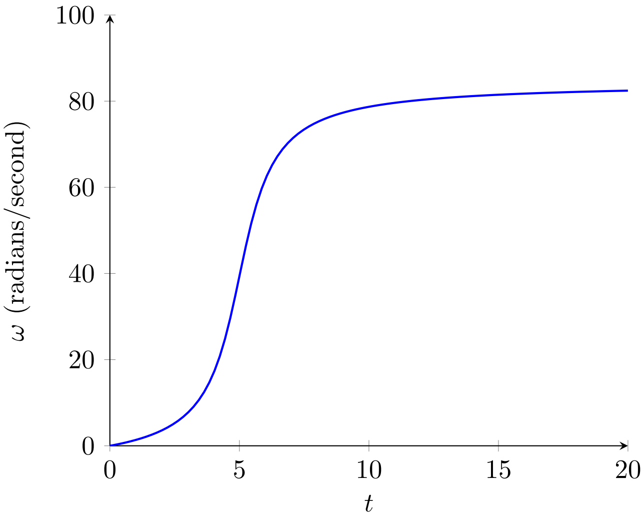


Figure 54: **Practice Problem:** A fan begins at rest and then starts to rotate faster and faster as shown in the Figure? What is the direction of $\vec{\omega}$ and $\vec{\alpha}$? What if the fan begins slowing down instead?

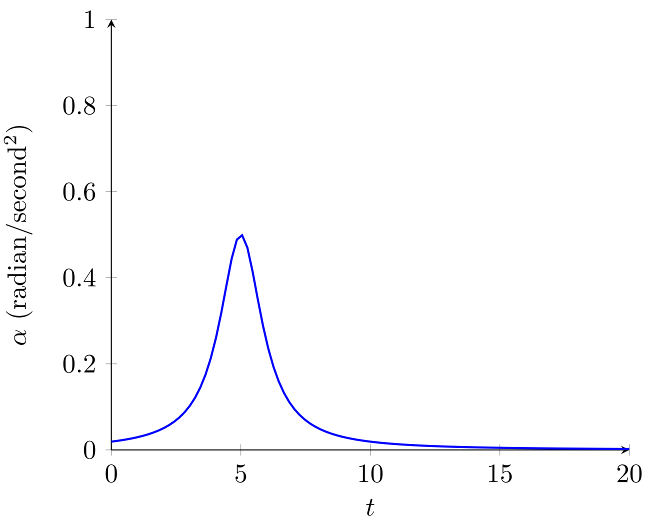
Solution: By the right-hand rule, $\vec{\omega}$ points out of the page. When the fan is speeding up, $\vec{\alpha}$ points in the same direction (out of the page). When the fan is slowing down, $\vec{\omega}$ still points out of the page, but $\vec{\alpha}$ points in the opposite direction into the page.

Practice Problem: Suppose a fan begins at rest and starts to spin, getting faster and faster until it reaches its maximum speed and continues spinning at the same speed. What does the graph of of ω versus time look like? What about the graph of α versus time?

Solution: See Figure 55.



(a)



(b)

Figure 55: Angular velocity and acceleration for a fan begins at rest and starts to spin, getting faster and faster until it reaches its maximum speed and continues spinning at the same speed.

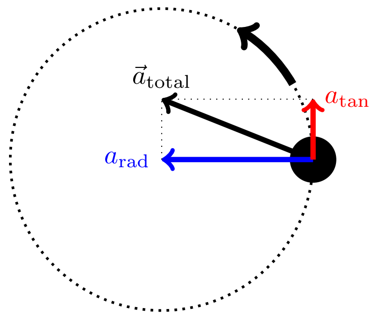


Figure 56: The tangential acceleration a_{tan} is the part of the total acceleration which is tangent to the object's path. The radial acceleration a_{rad} is the part which is perpendicular to the object's path.

16.3.1 Tangential Acceleration

We have to be careful because $\frac{d}{dt}|\vec{v}|$ is not the same as the linear acceleration $a = \frac{d}{dt}\vec{v}$. Instead, $\frac{d}{dt}|\vec{v}| = a_{\text{tan}}$ is the *tangential* acceleration. This is the component of $\vec{a}$ that is parallel to $\vec{v}$ (and tangent to the object's path; see Figure 56). In terms of a_{tan},

$$|\vec{\alpha}| = \frac{a_{\text{tan}}}{R}.$$

Rearranging the equation,

$$\boxed{a_{\text{tan}} = R|\vec{\alpha}|.}$$

16.3.2 Radial Acceleration

The other component of the linear acceleration is the radial component (the part of the acceleration that is perpendicular to the object's circular path; see Figure 56). If the object is traveling in a circle, then the radial acceleration is given by the formula we derived earlier for the special case of uniform circular motion:

$$\boxed{a_{\text{rad}} = \frac{|\vec{v}|^2}{R}.}$$

16.4 Constant Angular Acceleration

We have introduced three kinematic variables for rotational motion. Figure 57 summarizes the rotational kinematic variables next to their linear counterparts. Earlier, when we studied linear motion, we derived the equations that apply for problems with constant acceleration. By the same process, we can derive very similar equations when there is constant *angular* acceleration. These equations are summarized in Figure 58 next to their linear counterparts. We can use these equations for problems with constant angular acceleration just like we earlier used the linear versions of these equations for problems with constant acceleration.

Practice Problem: A car, which was initially at rest, is traveling at a constant acceleration (which means its wheels have a constant angular acceleration). After 2 seconds, it has traveled 20 m. The radius of the wheels is 0.5 m. What is the angular acceleration of the wheel? Assume the wheels turn without slipping. Hint: figure out how many radians the wheels have gone through after 2 seconds.

Solution: When the wheels go around one time, the car will have traveled $2\pi R \approx 3.14$ m. Since the car traveled 20 m, the wheels must have completed $\frac{20 \text{ m}}{3.14 \text{ m}} \approx 3.67$ complete revolutions. Each revolution involved 360° (which is 2π radians). Therefore, the wheels have gone through $2\pi \cdot 3.67 \approx 40$ radians in 2 seconds. Now we can use the kinematic equations. Choose the initial angle of the wheels to be zero radians. Then $\theta_0 = \omega_0 = 0$ and

$$\theta = \theta_0 + t\omega_0 + \frac{1}{2}t^2\alpha$$

$$\implies 40 \text{ rad} = 0 + 0 + \frac{1}{2}(2 \text{ s})^2\alpha$$

$$\implies \alpha = 2 \cdot \frac{1}{4 \text{ s}^2} \cdot 40 \text{ rad} = 20 \text{ rad/s}^2.$$

Rotational	Linear
θ	$\vec{x}$
$\lvert\vec{\omega}\rvert = \frac{d\theta}{dt}$	$\vec{v} = \frac{d\vec{x}}{dt}$
$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$	$\vec{a} = \frac{d\vec{v}}{dt}$

Figure 57: The rotational kinematic variables and their linear counterparts.

Rotational	Linear
$\vec{\alpha}(t) = \text{constant}$	$\vec{a}(t) = \text{constant}$
$\vec{\omega}(t) = \vec{\omega}_0 + t\vec{\alpha}$	$\vec{v}(t) = \vec{v}_0 + t\vec{a}$
$\theta(t) = \theta_0 + t\omega_0 + \frac{1}{2}t^2\alpha$	$\vec{x}(t) = \vec{x}_0 + t\vec{v}_0 + \frac{1}{2}t^2\vec{a}$
$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$	$v^2 = v_0^2 + 2a\Delta x$

Figure 58: The kinematic equations for problems with constant rotational/linear acceleration.

16.5 Moment of Inertia and Rotational Kinetic Energy

Rotation contributes to an object's kinetic energy. For linear motion, the kinetic energy was given by the formula

$$K = \frac{1}{2}mv^2.$$

For an object in rotational motion, we can use the formula $v = R\omega$ to get

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(R\omega)^2 = \frac{1}{2}mR^2\omega^2.$$

We define the **moment of inertia** I of a point mass moving in a circle of radius R by the equation

$$\boxed{I = mR^2.}$$

Then the kinetic energy of an object in rotational motion is

$$\boxed{K = \frac{1}{2}I\omega^2.}$$

If there are multiple point masses, all connected so they are moving with the same angular velocity, then

$$K = \sum_i \frac{1}{2}m_i v_i^2 = \sum_i \frac{1}{2}m_i(R_i\omega)^2 = \frac{1}{2}\left(\sum_i m_i R_i^2\right)\omega^2.$$

Therefore, if there are multiple connected point masses, we define

$$\boxed{I = \sum_i m_i R_i^2}$$

and the kinetic energy is still

$$K = \frac{1}{2}I\omega^2.$$

17 Moment of Inertia

17.1 Definition

During previous classes, we learned
We found that the moment of inertia I (which appears in the rotational kinetic energy formula) is given by

$$\boxed{I = \sum_i m_i |r_i|^2.}$$

$\sum_i$ is a sum over all of the different masses in the system, and $|r_i|$ is the distance to each mass from the axis of rotation.

Rotational	Translational
θ	$\vec{x}$
$\omega = \frac{d\theta}{dt}$	$\vec{v} = \frac{d\vec{x}}{dt}$
$\alpha = \frac{d\omega}{dt}$	$\vec{a} = \frac{d\vec{v}}{dt}$
$K_{\text{rot}} = \frac{1}{2}I\omega^2$	$K_{\text{tr}} = \frac{1}{2}mv^2$

Figure 59: Rotational variables versus translational variables.

17.1.1 Example

Suppose there are two masses $M_1 = 0.2$ kg and $M_2 = 0.1$ kg, rotating around the center of a disk, as shown in Figure 60. M_1 is a distance of 0.85 m from the center of the disk, and M_2 is a distance of 0.64 m from the center of the disk. If the disk is rotating with an angular speed of 0.6 rad/s, what is the moment of inertia of the two masses? What is the rotational kinetic energy of the two masses? (Ignore the moment of inertia and rotational kinetic energy of the disk: we only want the kinetic energy of the two masses).

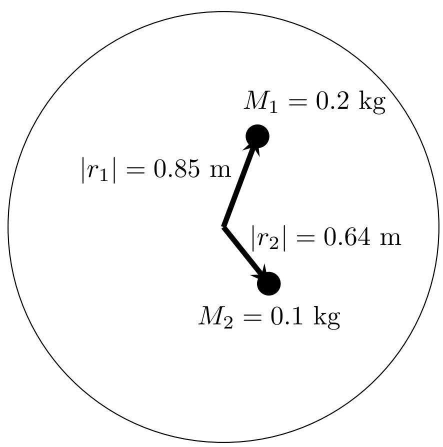


Figure 60: Two point masses are rotating around the center of a disk.

Solution:
The moment of inertia of the two masses is

$$I = \sum_i m_i |r_i|^2 = M_1|r_1|^2 + M_2|r_2|^2 = 0.2 \text{ kg} \cdot (0.85 \text{ m})^2 + 0.1 \text{ kg} \cdot (0.64 \text{ m})^2$$

$$\approx 0.185 \text{ kg} \cdot \text{m}^2.$$

The rotational kinetic energy is therefore

$$K_{\text{rot}} = \frac{1}{2}I\omega^2 = \frac{1}{2}(0.185\ \text{kg}\cdot\text{m}^2)(0.6\ \text{rad/s})^2 \approx 0.033\ \text{kg}\cdot\text{m/s}^2 = 0.033\ \text{J}.$$

Remember that radians are dimensionless, which means we can simply ignore them when combining units.

Note: The moment of inertia depends on the axis of rotation! If I instead had said that the two masses were rotating around, for example, the center of M_1, then the moment of inertia would be different. Always pay attention to the location of the axis of rotation.

17.2 Intuition

Based on the formula

$$I = \sum_i m_i |r_i|^2,$$

masses that are further away from the axis of rotation contribute significantly more to the moment of inertia.

17.2.1 Example

Consider a disk and a ring with the same radius and same total mass, as shown in Figure 61. Which has a higher moment of inertia (around the axis that goes straight out of the page through its center)?

Figure 61: A disk and a ring of equal radius and total mass.

Solution:
The ring has a higher moment of inertia because all the mass is concentrated far away from the axis of rotation. Based on the formula $I = \sum_i m_i |r_i|^2$, objects that have mass located far from the axis of rotation (large $|r_i|$) will have larger moments of inertia.

17.2.2 Example

Consider two rockets of equal mass. One has most of the mass concentrated in its center, and one has most of the mass concentrated at the top. Which rocket has a higher moment of inertia for rotation around its center? Which rocket has a higher moment of inertia for rotation around the top?

Solution: The rocket with mass concentrated at the top has a higher moment of inertia for rotation around its center, because most of the mass is concentrated at the top, far away from the center. However, the rocket with mass concentrated at the center has a higher moment of inertia for rotation around the top, because most of the mass is concentrated far away from the top.

17.3 Calculation

The formula

$$I = \sum_i m_i |r_i|^2$$

works if we have a collection of small point masses. However, it does not work for objects that are spread out, like a circle, ring, rod, sphere (or anything that isn't just a single point). To calculated the moment of inertia of these objects, we need to replace the sum with an integral:

$$I = \int |r|^2 dm.$$

You are not expected to do integrals in this class, so we will just give you the results of these calculations for various shapes when you need them. The moments of inertia of several shapes around different axes are listed in your equation sheet. Remember that the axis of rotation matters! The moments of inertia will be different depending on the axis!

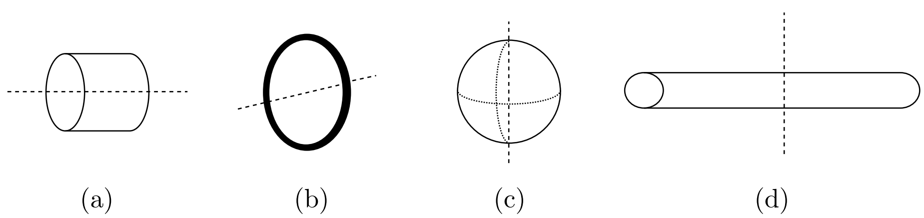


Figure 62: **Momenta of inertia for various shapes**
(a) Solid cylinder or disk, axis of rotation through the axis of the cylinder: $I = (1/2)MR^2$
(b) Thin ring, axis through the center and perpendicular to the plane of the ring: $I = MR^2$
(c) Solid sphere, axis through its center: $I = (2/5)MR^2$
(d) Thin rod, axis through center and perpendicular: $I = (1/12)ML^2$

17.4 Parallel Axis Theorem

The Parallel Axis Theorem allow you to calculate the moment of inertia around one axis based on the moment of inertia around a different, but parallel, axis.

Parallel Axis Theorem: If the moment of inertia about axis 1 is I_1, and if axis 2 is parallel to axis 1, then the moment of inertia about axis 2 is

$$\boxed{I_2 = I_1 + MD^2.}$$

M is the total mass of the system, and D is the distance between axis 1 and axis 2.

17.4.1 Example

Based on the equation sheet, the moment of inertia of a solid sphere about an axis through its center is

$$I_{\text{center}} = \frac{2}{5}MR^2$$

(M is the total mass of the sphere and R is the radius of the sphere). What is the moment of inertia of a solid sphere about a line tangent to the sphere?

Solution: We can use the parallel axis theorem. Axis 1 is a line through the center of the sphere. Axis 2 is a line tangent to the sphere. We can choose these lines so that they are parallel to each other. Then

$$I_{\text{tangent}} = I_{\text{center}} + MD^2.$$

In this case, the distance between axis 1 and axis 2 is just the radius of the sphere ($D = R$), and so

$$I_{\text{tangent}} = I_{\text{center}} + MD^2 = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2.$$

17.5 Superposition Principle

The Superposition Principle: When a system is composed of multiple objects, all rotating around the same axis, then the moment of inertia of the whole system can be found by adding the moments of inertia of all the parts of the system:

$$\boxed{I_{\text{total}} = \sum_i I_i.}$$

17.5.1 Example

Consider a bar of length $L = 3$ m with two solid balls of radius 0.2 m attached to its ends, as in Figure 63. The balls are attached so that their centers are 3 m apart. The bar has mass $m_{\text{bar}} = 2$ kg and the balls each have mass $m_{\text{ball}} = 10$ kg. What is the moment of inertia of the bar-and-balls system around an axis that is perpendicular to the bar and passes through the center of the bar?

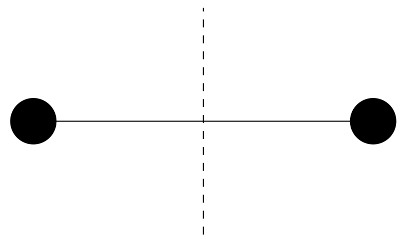


Figure 63: A bar with two solid balls attached to its ends. The dashed line is the axis of rotation.

Solution: We will use the superposition principle:

$$I_{\text{total}} = I_{\text{ball 1}} + I_{\text{bar}} + I_{\text{ball 2}}.$$

Based on the equation sheet, the moment of inertia of a bar around the given axis is

$$I_{\text{bar}} = \frac{1}{12} m_{\text{bar}} L^2 = \frac{1}{12}(2\ \text{kg})(3\ \text{m})^2 = 1.5\ \text{kg} \cdot \text{m}^2.$$

Based on the equation sheet, the moment of inertia of a solid ball around its center is $\frac{2}{5} m_{\text{ball}} R^2$. However, in this case, the axis of rotation is not through the center of the balls. We need to use the parallel axis theorem to move the axis of rotation from the center of each all to the center of the bar. The distance from the center of each ball to the center of the bar is $\frac{L}{2}$. Therefore, by the parallel axis theorem,

$$I_{\text{ball}} = \frac{2}{5} m_{\text{ball}} R^2 + m_{\text{ball}} \left(\frac{L}{2}\right)^2 = \frac{2}{5}(10\ \text{kg})(0.2\ \text{m})^2 + (10\ \text{kg}) \left(\frac{3\ \text{m}}{2}\right)^2$$
$$= 23.5\ \text{kg} \cdot \text{m}^2.$$

The moment of inertia is the same for both balls in this case: both of their axes need to be moved by a distance of $\frac{L}{2}$. The total moment of inertia is therefore

$$I_{\text{total}} = I_{\text{ball 1}} + I_{\text{bar}} + I_{\text{ball 2}} = 23.5\ \text{kg}\cdot\text{m}^2 + 1.5\ \text{kg}\cdot\text{m}^2 + 23.5\ \text{kg}\cdot\text{m}^2 = 48.5\ \text{kg}\cdot\text{m}^2.$$

17.6 Total Kinetic Energy

When an object is both moving *and* rotating, its kinetic energy is given by

$$\boxed{K_{\text{total}} = K_{\text{tr,CM}} + K_{\text{rot,CM}}.}$$

$K_{\text{tr,CM}}$ is the translational kinetic energy of the center of mass, defined by $K_{\text{tr,CM}} = \frac{1}{2} M |v_{\text{CM}}|^2$, where M is the total mass and v_{CM} is the velocity of the center of mass. $K_{\text{rot,CM}}$ is the rotational kinetic energy around the center of mass, defined by $\frac{1}{2} I_{\text{CM}} \omega^2$, where I_{CM} is the moment of inertia about the center of mass.

17.6.1 Example

A ball of radius 0.05 m and mass 0.2 kg is thrown at a velocity of 40 m/s and no spin. A second ball (of equal radius and mass) is thrown with a velocity of 40 m/s and spins at a rate of 200 radians per second. How much more kinetic energy does the second ball have?

Solution: The translational kinetic energies of the first and the second ball are the same: $K_{\text{tr,CM}} = \frac{1}{2}M|v_{\text{CM}}|^2 = \frac{1}{2}(0.2\text{ kg})(40\text{ m/s})^2$. The only difference is the rotational kinetic energy. The first ball has no rotational kinetic energy:

$$K_{\text{rot,CM,ball 1}} = \frac{1}{2}I_{\text{CM}}\omega^2 = \frac{1}{2}I_{\text{CM}} \cdot 0 = 0.$$

The rotational kinetic energy of the second ball is

$$K_{\text{rot,CM,ball 1}} = \frac{1}{2}I_{\text{CM}}\omega^2 = \frac{1}{2}\left(\frac{2}{5}m_{\text{ball}}R^2\right)\omega^2 = \frac{1}{2}\left(\frac{2}{5}(0.2\text{ kg})(0.05\text{ m})^2\right)(200\text{ rad}/s)^2$$
$$= 2\text{ kg}\cdot\text{m}^2/\text{s}^2 = 2\text{ J}.$$

Therefore, the second ball has 2 J more kinetic energy than the first.

18 Torque

18.1 The Cross Product

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is written as $\vec{A}\times\vec{B}$. This is not the same as the dot product $\vec{A}\cdot\vec{B}$. The cross product gives a vector, not just a number. The magnitude of the cross-product is given by

$$\boxed{|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta,}$$

where θ is the angle between $\vec{A}$ and $\vec{B}$. For this formula, you should always choose the **smaller** angle between $\vec{A}$ and $\vec{B}$ (see Figure 64 for an example). Otherwise, you will end up with a negative sign. The direction of the cross product is given by the right-hand rule:

Right-hand rule (for cross product): Point your right thumb along $\vec{A}$ and your other fingers along $\vec{B}$. Then your palm points in the direction of $\vec{A}\times\vec{B}$. The order is important here. If you use the wrong order, you will get the opposite direction!

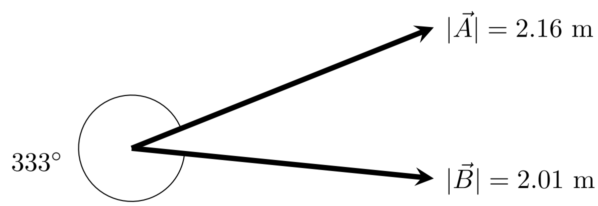


Figure 64: The cross product $\vec{A} \times \vec{B}$ has magnitude $|\vec{A}||\vec{B}|\sin(360° - 333°) = 2.16 \cdot 2.01 \sin(27°)\ \mathrm{m}^2 \approx 1.97\ \mathrm{m}^2$. By the right-hand rule, its direction is into the page (the only way to get your thumb to point along $\vec{A}$ and your fingers to point along $\vec{B}$ is to point your palm toward the page).

Cross product	**Dot product**
$\vec{A} \times \vec{B}$ is a vector	$\vec{A} \cdot \vec{B}$ is a number
$\lvert\vec{A} \times \vec{B}\rvert = \lvert\vec{A}\rvert\lvert\vec{B}\rvert \sin\theta$	$\vec{A} \cdot \vec{B} = \lvert\vec{A}\rvert\lvert\vec{B}\rvert \cos\theta$
Direction of $\vec{A} \times \vec{B}$ given by right-hand rule	$\vec{A} \cdot \vec{B}$ has no direction

Figure 65: The dot product versus the cross product.

18.1.1 Special Cases

- If $\vec{A}$ and $\vec{B}$ are parallel ($\theta = 0$) or anti-parallel ($\theta = 180°$), then the cross product is zero ($\sin 0 = \sin 180° = 0$).
- If $\vec{A}$ and $\vec{B}$ are perpendicular ($\theta = 90°$), then the cross product has its **maximum** magnitude of $|\vec{A}||\vec{B}|$ (because $\sin 90° = 1$).
- The order of the cross product matters. When you switch $\vec{A}$ and $\vec{B}$, the cross product will be in the opposite direction:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}.$$

18.1.2 3D Vector Notation

We will use the symbol $\odot$ to represent a vector coming *out of* the page, and the symbol $\otimes$ to represent a vector *going into* the page.

$\vec{\mathbf{A}}$	$\vec{\mathbf{B}}$	$\vec{\mathbf{A}} \times \vec{\mathbf{B}}$
$\searrow$	$\nearrow$	$\odot$
$\odot$	$\nearrow$	$\nwarrow$
$\swarrow$	$\otimes$	$\searrow$
$\swarrow$	$\nwarrow$	$\otimes$
$\odot$	$\otimes$	This is zero (because the angle between $\vec{A}$ and $\vec{B}$ is 180°)

Figure 66: The direction of the cross product of various pairs of vectors.

18.2 Torque Definition

During previous classes, we've made a side-by-side comparison of rotational and translational variables

Rotational	Translational
θ	$\vec{x}$
$\omega = \frac{d\theta}{dt}$	$\vec{v} = \frac{d\vec{x}}{dt}$
$\alpha = \frac{d\omega}{dt}$	$\vec{a} = \frac{d\vec{v}}{dt}$
$K_{\text{rot}} = \frac{1}{2} I\omega^2$	$K_{\text{tr}} = \frac{1}{2} mv^2$

Figure 67: Rotational variables versus translational variables.

In this class, we will add one more row to this table for Newton's second law. For translational variables, Newton's second law is $\sum_i \vec{F}_i = m\vec{a}$. In rotational variables, Newton's second law becomes:

$$\boxed{\sum_i \vec{\tau}_i = I\vec{\alpha}.}$$

The mass m has been replaced by the moment of inertia I, the acceleration a has been replaced with the angular acceleration α, and the sum of forces $\sum_i \vec{F}_i$

has been replaced with a sum of torques $\sum_i \tau_i$. Torque is defined by

$$\boxed{\tau_i = \vec{r}_i \times \vec{F}_i.}$$

In this formula, $\vec{F}_i$ is the force that is causing the torque, and $\vec{r}_i$ is the vector that points from the axis of rotation to the location where the force is being applied. The vector $\vec{r}_i$ is called the **lever arm**.

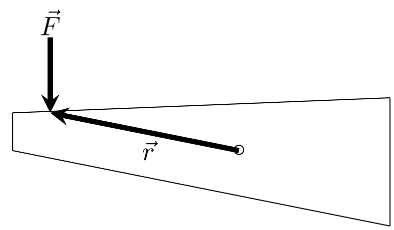


Figure 68: The axis of rotation is marked with $\circ$. The force $\vec{F}$ is applied at the location shown in the figure. The lever arm $\vec{r}$ points from the axis of rotation to the point where the force is applied. The torque is $\vec{r} \times \vec{F}$ and points out of the page.

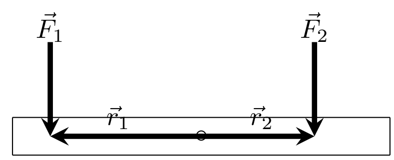


Figure 69: $\vec{F}_1$ produces a torque pointing out of the page, while $\vec{F}_2$ produces a torque pointing into the page.

Rotational	Translational
θ	$\vec{x}$
$\omega = \frac{d\theta}{dt}$	$\vec{v} = \frac{d\vec{x}}{dt}$
$\alpha = \frac{d\omega}{dt}$	$\vec{a} = \frac{d\vec{v}}{dt}$
$K_{\text{rot}} = \frac{1}{2} I \omega^2$	$K_{\text{tr}} = \frac{1}{2} m v^2$
$\sum_i \vec{\tau}_i = I \vec{\alpha}$	$\sum_i \vec{F}_i = m \vec{a}$

Figure 70: Rotational variables versus translational variables, including Newton's second law.

18.3 Working with Torque

If we use the symbol $\hat{n}_i$ to represent the direction of each torque, then we can write the torque formula as

$$\tau_{\text{total}} = \sum_i \vec{r}_i \times \vec{F}_i = \sum_i |\vec{r}_i||\vec{F}_i| \sin\theta_i \cdot \hat{n}_i.$$

Based on this formula,

- forces that are applied further away from the axis of rotation will create more torque (for a given $|\vec{F}|$ and θ),
- forces that are perpendicular ($\theta = 90°$) to the lever arm will create more torque (for a given $|\vec{F}|$ and $|\vec{r}|$), and
- forces that are parallel ($\theta = 0°$) or anti-parallel ($\theta = 180°$)t o the lever arm will **not** cause any torque.

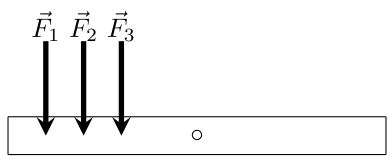


Figure 71: For a plank rotating around its center as shown, all three of the applied forces produce a torque pointing out of the page, which means the angular acceleration will point into the page. $\vec{F}_1$ produces the torque with the largest magnitude, since it is farthest from the axis of rotation, and $\vec{F}_3$ produces the torque with the smallest magnitude, since it is closest to the axis of rotation.

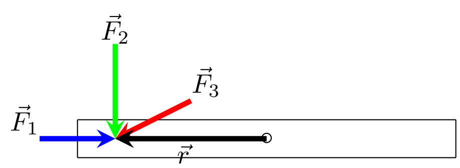


Figure 72: Three forces of equal magnitude are applied to a plank rotating around its center as shown. Even though each forces has the same magnitude and is applied at the same point, they will all produce a different torque because they all make different angles with the lever arm $\vec{r}$. $\vec{F}_1$ will produce no torque because it is anti-parallel to $\vec{r}$. $\vec{F}_2$ will produce the most torque, since it is perpendicular to $\vec{r}$.